University of Graz
Institute of Mathematics and Scientific Computing

Dr. Silvio Fanzon
silvio.fanzon@uni-graz.at

Inverse Problems - Exercise Sheet 6

Publication date: 12 January, 2023

Due date: 24 January, 2023

Exercise 6.1 (20 pts) - Tikhonov regularization with semi-norms. Let X, Y, and Z be Banach spaces, where X is additionally reflexive and Z is separable. Further, let $T : X \to Y$ be a linear and continuous operator, and $A : \mathrm{dom}(A) \to Z^*$ be a densely defined, linear operator on the Banach space X. Moreover, assume that the range of A is closed in Z^*, the kernel of A is finite-dimensional, and that A is weakly-weakly$*$ closed, which means that $x_n \rightharpoonup x$ and $Ax_n \overset{*}{\rightharpoonup} z$ implies $x \in \mathrm{dom}(A)$ and $Ax = z$.
For each $y^\delta \in Y, p, q \geq 1$, and $\alpha > 0$, show that there exists a solution of the minimization problem

$$\min_{x \in X} \frac{1}{q} \left\| Tx - y^\delta \right\|_Y^q + \frac{\alpha}{p} \left\| Ax \right\|_{Z^*}^p .$$

Hint: You may exploit: if the linear operator $S : \mathrm{dom}(S) \to Y$ between the dense subspace $\mathrm{dom}(S) \subset X$ and Y is closed and bijective, then S^{-1} is continuous. Further, note that the finite-dimensional subspace $\ker(A)$ is complemented in X.

Convex analysis. Let X be a Banach space, X^* its dual, and $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ a function. We define the convex conjugate of f as the function $f^* : X^* \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$f^*(x^*) = \sup_{x \in X} \ \langle x^*, x \rangle - f(x) , \tag{1}$$

where $\langle x^*, x \rangle$ denotes the dual pairing between X and X^*. The convex conjugate satisfies the following properties:

- f^* is always convex and lower-semicontinuous.
- The convex biconjugate f^{**} (that is to apply the convex conjugate twice), corresponds to the closed convex hull of f, i.e. the largest lower semi-continuous convex function such that $f^{**} \leq f$. If f is proper, convex and lower semicontinuous, then $f^{**} = f$.
- **Fenchel duality:** For $f \colon X \to \mathbb{R} \cup \{\pm\infty\}$, $g \colon Y \to \mathbb{R} \cup \{\pm\infty\}$ convex functions and $A \colon X \to Y$ bounded linear operator, the following inequality holds

  $$\inf_{x \in X} \ f(Ax) + g(x) \geq \sup_{y^* \in Y^*} \ -f^*(-y^*) - g^*(A^* y^*) .$$

  Equality holds under certain regularity conditions. The above inequality is also named the weak duality of the *primal* and *dual* optimization problems. In the case of equality of the two optimization problems, it is said that there is strong duality.

Exercise 6.2 (30 pts) - The convex conjugate.

a) Consider a Banach space X with norm $\|\cdot\|_X$. Prove that the convex conjugate (1) of the norm is the convex indicator function over the unitary ball in the dual space, that is,

$$\left(\|\cdot\|_X\right)^* (x^*) = \mathbb{1}_{B^*}(x^*) = \begin{cases} 0 & \text{if} \quad \|x^*\|_{X^*} \leq 1, \\ \infty & \text{if} \quad \|x^*\|_{X^*} > 1 . \end{cases}$$

b) Consider the space $X = \mathbb{R}^n \times \mathbb{R}^n$ endowed with the norm $||x|| = \sum_{i=1}^n \sqrt{y_i^2 + z_i^2}$, for $x = (y, z)$. Prove that for $x^* = (y^*, z^*) \in X$ one has

$$(|| \cdot ||)^* \, (x^*) = \begin{cases} 0 & \text{if} \quad \max\limits_i \sqrt{(y_i^*)^2 + (z_i^*)^2} \leq 1, \\ \infty & \text{if} \quad \max\limits_i \sqrt{(y_i^*)^2 + (z_i^*)^2} > 1, \end{cases} \tag{2}$$

The primal-dual algorithm. We want to compute a solution to the problem

$$\min_{x \in X} \; f(Ax) + g(x) \,, \tag{3}$$

where:

- X, Y are two finite-dimensional real vector spaces equipped with an inner product $\langle \cdot, \cdot \rangle$ and norm.
- $A : X \to Y$ is a continuous linear operator
- $g : X \to [0, \infty)$ and $f : Y \to [0, \infty)$ are proper, convex, lower-semicontinuous functions.

For a function h the *resolvent operator* is defined as

$$(I + \tau \partial h)^{-1} \, (y) = \arg\min_x \left\{ \frac{\|x - y\|^2}{2\tau} + h(x) \right\} .$$

Then, for selected constants $\tau, \sigma > 0, \theta \in [0, 1]$, initial guess $(x^0, y^0) \in X \times Y$, f^* the convex conjugate of f, and A^* the adjoint operator of A, the *primal-dual algorithm* stopped at N iterations is

Algorithm 1 Primal-dual Algorithm

procedure PRIMAL-DUAL$(\tau, \sigma, \theta, N, x^0, y^0)$
    $\overline{x}^0 = x^0$
    for $n = 0, 1, 2, \ldots, N$ **do**
        $y^{n+1} = (I + \sigma \partial f^*)^{-1} \, (y^n + \sigma A \overline{x}^n)$
        $x^{n+1} = (I + \tau \partial g)^{-1} \, (x^n - \tau A^* y^{n+1})$
        $\overline{x}^{n+1} = x^{n+1} + \theta (x^{n+1} - x^n)$
    end for
    return x^{N+1}.
end procedure

The ROF model: We now introduce the ROF model for image denoising. We model a 2D image u by a $N \times N$ matrix, $N > 0$ integer. Thus $u \in \mathbb{R}^{N\times N}$. The discrete gradient of u with Neumann boundary conditions is defined by $\nabla u = (\nabla_x u, \nabla_y u)$, where $\nabla_x u, \nabla_y u \in \mathbb{R}^{N\times N}$ are matrices defined by

$$(\nabla_x u)_{i,j} := \begin{cases} u_{i+1,j} - u_{i,j} & \text{if} \quad i < N-1\,, \\ 0 & \text{if} \quad i = N-1\,, \end{cases} \qquad (\nabla_y u)_{i,j} := \begin{cases} u_{i,j+1} - u_{i,j} & \text{if} \quad j < N-1\,, \\ 0 & \text{if} \quad j = N-1\,. \end{cases} \tag{4}$$

For $u \in \mathbb{R}^{N\times N}$, $p = (p^x, p^y)$, $p^x, p^y \in \mathbb{R}^{N\times N}$ we define

$$\|u\|_2 := \sqrt{\sum_{i,j=1}^{N} u_{i,j}^2}\,, \qquad \|p\|_{1,2} := \|p^x\|_2 + \|p^y\|_2\,. \tag{5}$$

Given a noisy image $D \in \mathbb{R}^{N\times N}$, the denoised image $u \in \mathbb{R}^{N\times N}$ according to the ROF model is a solution to the minimization problem

$$\min_{u\in\mathbb{R}^{N\times N}} \|\nabla u\|_{1,2} + \frac{\lambda}{2}\|u - D\|_2^2\,, \tag{6}$$

where $\lambda > 0$ is a fixed regularization parameter.

Exercise 6.3 (50 pts) - Implementing the primal-dual algorithm for ROF. In this exercise we will implement the primal-dual algorithm to compute solutions of the ROF model for image denoising at (6). We define the following:

- $X := \mathbb{R}^{N\times N}$ normed by $\|\cdot\|_X := \|\cdot\|_2$ as defined in (5). Notice that such norm is induced by the Hilbert product
$$\langle u, v\rangle_X := \langle u, v\rangle_2 := \sum_{i,j=1}^{N} u_{i,j} v_{i,j}\,,$$
for $u, v \in X$.
- $Y := \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times N}$ normed by $\|\cdot\|_Y := \|\cdot\|_{1,2}$ as defined in (5). This norm is induced by the Hilbert product
$$\langle p, q\rangle_Y := \langle p, q\rangle_{1,2} := \langle p^x, q^x\rangle_2 + \langle p^y, q^y\rangle_2$$
for $p = (p^x, p^y), q = (q^x, q^y) \in Y$.
- $A\colon X \to Y$ linear continuous operator defined by $Au := \nabla u$ where $\nabla = (\nabla_x, \nabla_y)$ is as in (4).
- $f : Y \to [0, \infty)$ defined by $f(p) := \|p\|_{1,2}$.
- Given $\lambda > 0$ and $D \in X$, define $g_\lambda : X \to [0, \infty)$ as $g_\lambda(u) := \frac{\lambda}{2}\|u - D\|_2^2$.

With the above definitions, please address the following questions.

a) Compute A^*, the adjoint of the operator A.

b) Following the provided template `primal_dual.py`, implement the operators A and A^*.

c) By Exercise 6.2 we have that

$$f^*(p) = \begin{cases} 0 & \text{if} \quad \max_{i,j}\sqrt{(p_{i,j}^x)^2 + (p_{i,j}^y)^2} \le 1, \\ \infty & \text{if} \quad \max_{i,j}\sqrt{(p_{i,j}^x)^2 + (p_{i,j}^y)^2} > 1. \end{cases}$$

Dr. Silvio Fanzon
silvio.fanzon@uni-graz.at

Prove that the resolvent of f^* has the following form:

$$p = (I + \sigma \partial f^*)^{-1} (\tilde{p}) \qquad \Longleftrightarrow \qquad p_{i,j} = \frac{\tilde{p}_{i,j}}{\max\left(1, \sqrt{(\tilde{p}^x_{i,j})^2 + (\tilde{p}^y_{i,j})^2}\right)} .$$

d) Following the provided template `primal_dual.py`, implement the resolvent of f^*.

e) Prove that the resolvent of g_λ has the following form:

$$(I + \tau \partial g_\lambda)^{-1} (\tilde{u}) = \frac{\tilde{u} + \lambda \tau D}{1 + \lambda \tau} .$$

f) Following the provided template `primal_dual.py`, implement the resolvent of g_λ.

g) Following the provided template `primal_dual.py`, implement the primal-dual algorithm. Then run the code to obtain reconstructions for two noise cases. Feel free to change the parameters to see different results (λ, the number of iterations, noise level). Changing the values of σ and τ is discouraged, as it is related to the convergence properties of this algorithm, topic not discussed in this exercise.