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Inverse Problems - Exercise Sheet 5

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Exercise 5.1 (50 pts) - Singular values of the Radon transform For this exercise we fix the weight function $w(s) = \sqrt{1-s^2}$. Furthermore, we will require the Chebyshev polynomials

$$U_m(s) = \frac{\sin[(m+1)\arccos(s)]}{\sin(\arccos(s))}.$$

They satisfy the following properties:

- They are orthogonal in the $L^2([-1,1], w)$, the $w-$ weighted space. That is

$$\int_{-1}^{1} w(s) U_m(s) U_{m'}(s) ds = \frac{\pi}{2}\delta_{m,m'}.$$

- They form an orthogonal basis for $L^2([-1,1], w)$. Consequently, the functions $U_m(s)w^{-1}(s)$ form a basis of $Z = L^2([-1,1], w^{-1})$.
- They satisfy the following integral property

$$\int_{-w(s)}^{w(s)} U_m\left(\mathbf{w}(\theta') \cdot (s\mathbf{w}(\theta) + t\mathbf{w}^{\perp}(\theta))\right) dt = \frac{2}{m+1} \frac{\sin((m+1)(\theta-\theta'))\sin((m+1)\arccos(s))}{\sin(\theta-\theta')}$$

a) Prove that the Radon transform is continuous from $L^2(B)$ to $Z = L^2(\Omega, w(s)^{-1})$, the L^2 space weighted with w^{-1}, that is, show the following inequality

$$\|\mathcal{R}f\|_{L^2(\Omega, w^{-1})} = \int_0^{2\pi}\int_{-1}^{1} |\mathcal{R}[f](\theta, s)|^2 w(s)^{-1} ds d\theta \leq 4\pi \|f\|_{L^2(B)}$$

Hint 1: Notice that the interval $[-w(s), w(s)]$ corresponds to the effective interval in which the Radon transform integrates, for any direction θ and displacement s.
Hint 2: To integrate, reuse the same idea presented in the last exercise sheet.

b) Under this new metric on the image space of the Radon transform, find the adjoint operator $\mathcal{R}^*$.
Hint: you can use the results from the previous exercise class.

c) Consider functions of the form $g(s,\theta) = U(s)w(s)v(\theta)$, for $U \in L^2([-1,1]),\ v \in L^2([0,2\pi))$. Check that

$$(\mathcal{R}\mathcal{R}^* g)(\theta, s) = \int_{-w(s)}^{w(s)}\int_0^{2\pi} U\left(\mathbf{w}(\theta') \cdot (s\mathbf{w}(\theta) + t\mathbf{w}^{\perp}(\theta))\right) v(\theta') d\theta' dt$$

d) Check that

$$(\mathcal{R}\mathcal{R}^* g_m)(\theta, s) = \frac{4\pi}{m+1} w(s) U_m(s) \overline{v}(\theta),$$

where U_m are the Chebyshev polynomials, and $\overline{v}(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin((m+1)(\theta-\theta'))}{\sin(\theta-\theta')} v(\theta') d\theta'$.

e) Consider the family $Y_l(\theta) = \frac{1}{\sqrt{2\pi}} e^{-il\theta}$ of functions that form a basis for $L^2([0, 2\pi))$ and satisfy

$$\overline{Y}_l = \begin{cases} Y_l & \text{if } -m \leq l \leq m \text{ and } l - m \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Write a basis for Z. From it, find the eigenfunctions and nonzero eigenvectors of $\mathcal{R}\mathcal{R}^*$.

f) Obtain the singular values for the Radon transform $\mathcal{R}$. How ill-posed is the respective inverse problem? Why there seem to be many zero singular values?

Exercise 5.2 (50 pts) - Singular values of the Radon transform and the limited angle problem.

a) Write a code that returns the matrix associated to the Radon transform of a function with support in the unitary ball. For it use the provided template `RadonMatrix.py` (in Python, feel free to do an equivalent Matlab version). Take into consideration the following points

- The Radon transform goes from images in two dimensions to a function with two variables. Since it is written as a matrix, it will take vectors into vectors. Take care to be consistent in these transformations.
- The obtained matrix will be a $|S||\Theta| \times N^2$ sized matrix. It is recommended that you use a sparse format for it.
- The Radon transform is such that it acts only on objects that are supported in the unitary ball, so no weight should be given to pixels outside that area.
- Feel free to consider reasonable approximations. One suggested example could be to approximate the line integral by an uniform Riemann sum.
- Save the output results in the following sections, as computations could be a bit length (order of minutes).

b) In the template `RadonMatrix.py` there is a script to test the Matrix with an image composed of a non-centered disk, use it to check your implementation.

c) In the following points, fix $N = 40$, $S = 40$, $T = 60$. Use template `InvertData.py` to invert the operator for the cases $\theta \in [0, \pi], \theta \in [0, \pi/4]$, and noiseless and 8% of noise level. What do you observe when plotting the reconstructed images?

d) Write a script in which you plot the singular values (in a non-increasing fashion) of the matrix for the cases $\theta \in [0, 2\pi], [0, \pi], [0, \pi/2], [0, \pi/4]$. What do you see and what would it explain?

e) Write a script, in which for the case of $\theta \in [0, \pi/4]$ and 10% of noise, an approximate inversion is obtained by using all the singular values above: 0.001, 0.01, and 0.05. Comment on the results.