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Inverse Problems - Exercise Sheet 4

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Exercise 4.1 (50 pts) - Radon transform. For $f\colon B \to \mathbb{R}$ continuous, where $B := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, we define the Radon transform as

$$\mathcal{R}[f](\theta, s) = \int_{-\infty}^{\infty} f\left(s\ \mathbf{w}(\theta) + t\ \mathbf{w}^{\perp}(\theta)\right) dt, \quad (\theta, s) \in [0, 2\pi] \times \mathbb{R},$$

with f extended by 0 outside B and $\mathbf{w}(\theta) = (\cos(\theta), \sin(\theta))^t$, $\mathbf{w}^{\perp}(\theta) = (\text{-}\sin(\theta), \cos(\theta))^t$.

(a) Show that the Radon transform can be extended to a linear continuous operator from $L^2(B)$ into $L^2(Q)$ with $Q := [0, 2\pi] \times [-1, 1]$

(b) Prove that the adjoint of the Radon transform $\mathcal{R}^*$, also called **backprojection operator**, has the form

$$\mathcal{R}^*[g](x) = \int_0^{2\pi} g\left(\theta, x \cdot \mathbf{w}(\theta)\right) d\theta\,. \tag{1}$$

Note: Adjoint means that $\langle \mathcal{R}[f], g\rangle_{L^2(Q)} = \langle f, \mathcal{R}^*[g]\rangle_{L^2(B)}, \quad \forall f \in L^2(B), g \in L^2(Q)$.

Instructions for next exercise: Please bring your code on a pendrive on the day of the class

Exercise 4.2 (50 pts) - Numerical implementation of the Radon transform. We will now work in the discrete case where functions defined on $[-1, 1] \times [-1, 1]$ will be represented by square matrices.

(a) Using the provided template in Matlab `radontrans.m`, implement the Radon transform.

Hint: Consider using the `imrotate` function. If the code is taking more than 10 lines, reconsider your approach.

(b) The output of a Radon transform is called a sinogram, denoted by

$$g(\theta, s) = \mathcal{F}[f](\theta, s)\,.$$

From the sinogram at the left hand side of Figure 1, which could be the underlying source f that produces it?

(c) Using the provided template in Matlab `backprojection.m`, implement the adjoint of the Radon transform defined in (1).

Hint: Notice that for fixed θ, the function $G(x) = g(\theta, x \cdot \mathbf{w}(\theta)), x \in \mathbb{R}^2$, is constant along lines parallel to $\mathbf{w}^{\perp}(\theta)$.

Hint 2: Consider using the `imrotate` function. If the code is taking more than 10 lines, reconsider your approach.

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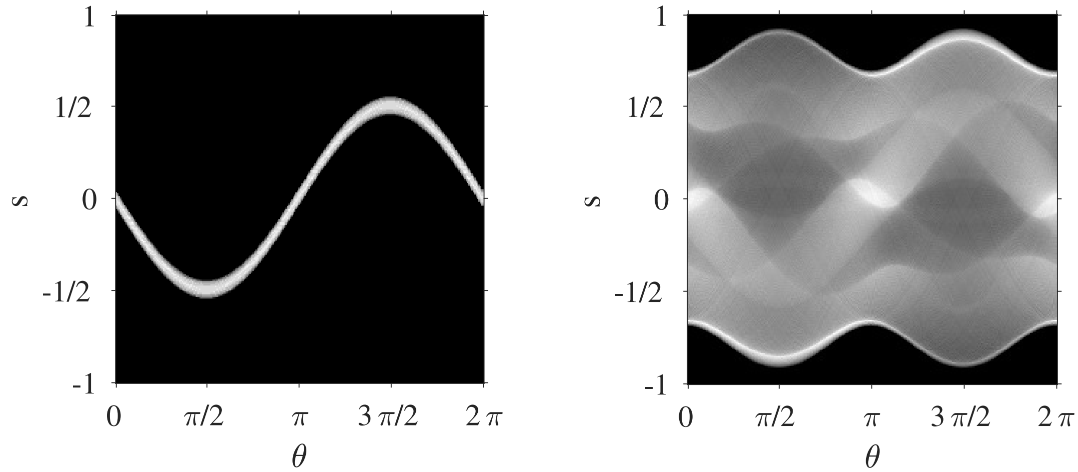

Figure 1: Example sinograms of unknown sources

(d) The **backprojection algorithm** corresponds to simply applying the adjoint operator $\mathcal{R}^*$ to the sinogram data. Write a script where you apply your implemented adjoint to the provided data `sinogram1.mat` and `sinogram2.mat`, that corresponds to the ones presented in Figure 1. What do you obtain?

Remark: It is required to use the same family of angles θ for the Radon transform and its adjoint. For the provided data $\theta =$ `linspace(0,360,256)`.

(e) Could you explain or interpret what is the backprojection algorithm $\mathcal{R}^*[\mathcal{R}[f]](x)$ doing?