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Inverse Problems - Exercise Sheet 3

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Exercise 3.1 (25 pts) - Compact operators are not coercive in infinite dimensions. Let X, Y be Banach spaces, $K \in \mathcal{L}(X, Y)$. Assume there exists a constant $c > 0$ such that

$$\|Kx\|_Y \geq c\,\|x\|_X \quad \text{for all} \quad x \in X\,. \tag{1}$$

Show that K is compact if and only if $\dim X < +\infty$.

Remark: An operator K satisfying (1) is called coercive. Notice that coercivity implies injectivity.

Exercise 3.2 (25 pts) - Non-injective compact operators form a dense set. Let X, Y be Banach spaces, with $\dim X = \infty$. Suppose that $K \in \mathcal{K}(X, Y)$

(a) Show that there exists a sequence $\{x_n\}_n$ in X such that $\|x_n\|_X = 1$ for all $n \in \mathbb{N}$ and $Kx_n \to 0$ strongly.

(b) Using (a), prove that the set

$$\{K \in \mathcal{K}(X, Y) : \; K \; \text{ is not injective } \}$$

is dense in $\mathcal{K}(X, Y)$ with respect to the operator norm.

Exercise 3.3 (25 pts) - Existence of minimal norm elements. Let X be a Banach space. Let $A \subset X$ be convex, closed and non-empty.

(a) Prove that the function

$$x \in X \mapsto \operatorname{dist}(x, A)$$

is lower semicontinuous with respect to the weak topology on X.

(b) Assume that X is reflexive. Prove that A has a minimal norm element, that is, show that there exists $\hat{x} \in A$ such that

$$\|\hat{x}\| = \inf_{x \in A} \|x\|$$

(c) Assume that X is reflexive. Let $M \subset X$ be a closed linear subspace. Prove that in Riesz's Lemma of Exercise 2.1 point (b) one can choose $\varepsilon = 0$, that is, show that there exists $x \in X$ such that

$$\|x\|_X = 1\,, \quad \operatorname{dist}(x, M) \geq 1\,.$$

Exercise 3.4 (25 pts) - Computation of Moore-Penrose Inverse For the following operators T in $\mathcal{L}(X, Y)$ compute the Moore-Penrose inverse of T, and check whether $\operatorname{Dom}(T^\dagger) = Y$.

(a) For Hilbert spaces X, Y and fixed $u \in X$ with $u \neq 0$ and $v \in Y$ with $v \neq 0$ let $T \colon X \to Y$ be defined according to $Tx = v\langle u, x\rangle$ (where $\langle\cdot,\cdot\rangle$ denote the scalar product on X).

(b) Let $T \colon L^2([0,1]) \to L^2([0,1])$ according to

$$[Tx](t) = \int_0^t x(s)\,ds \qquad \text{for almost every } t \in [0,1].$$