University of Graz
Institute of Mathematics and Scientific Computing

Dr. Silvio Fanzon
silvio.fanzon@uni-graz.at

Inverse Problems - Exercise Sheet 2

Publication date: 15 October, 2022 | Due date: 25 October, 2022

Motivation: compact operators often arise in inverse problems applications. In this worksheet we highlight a few of their properties.

Let X be a Banach space with norm $\|\cdot\|_X$ and denote by B_X its unit ball, that is,

$$B_X := \{x \in X \colon \|x\|_X \leq 1\} .$$

Let Y be a Banach space with norm $\|\cdot\|_Y$. We denote by $\mathcal{L}(X,Y)$ the space of linear continuous operators $K \colon X \to Y$. Recall that $\mathcal{L}(X,Y)$ is a Banach space with the operator norm. We set $\mathcal{L}(X) := \mathcal{L}(X,X)$. For a subset $M \subset X$ we denote the distance of $x \in X$ to M by

$$\operatorname{dist}(x, M) := \inf_{m \in M} \|x - m\|_X .$$

Compact operators: We say that $K \in \mathcal{L}(X,Y)$ is a *compact operator* if the closure of $K(B_X)$ is compact in Y. We denote the space of compact operators from X to Y by $\mathcal{K}(X,Y)$, and $\mathcal{K}(X) := \mathcal{K}(X,X)$.

Finite rank operators: We say that $K \in \mathcal{L}(X,Y)$ has *finite rank* if $K(X)$ is finite dimensional. Note that finite rank operators are clearly compact.

Adjoint: Let $K \in \mathcal{L}(X,Y)$. The *adjoint* of K is the linear operator $K^* \colon Y^* \to X^*$ defined by

$$\langle K^* y^*, \, x \rangle_{X^*,X} = \langle y^*, \, Kx \rangle_{Y^*,Y} \quad \text{for all} \quad x \in X, \; y^* \in Y^* .$$

It is well-known that $K^* \in \mathcal{L}(Y^*, X^*)$, with $\|K\| = \|K^*\|$.

Summable sequences: For $p \geq 1$ denote by ℓ_p the space of square summable sequences with norm $\|\cdot\|_{\ell_p}$, that is,

$$\ell_p := \left\{ x = (x_j)_{j \in \mathbb{N}} \, : \, \sum_{j=1}^{\infty} |x_j|^p < \infty \right\} , \quad \|x\|_{\ell_p} := \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} .$$

Recall that ℓ_2 is a Hilbert space with scalar product

$$\langle x, \, y \rangle_{\ell_2} := \sum_{j=1}^{\infty} x_j y_j .$$

Exercise 2.1 (30 pts) - The identity is not compact in infinite dimensions. Let X be a normed space. Suppose $M \subset X$ is a closed linear subspace with $M \neq X$.

(a) Let $x \in X \setminus M$. Prove that $\operatorname{dist}(x, M) > 0$.

(b) (Riesz's Lemma) Prove that for all $\varepsilon > 0$ there exists $x \in X$ such that

$$\|x\|_X = 1 , \quad \operatorname{dist}(x, M) \geq 1 - \varepsilon .$$

(c) Suppose in addition that X is a Banach space. Prove that the identity map $I \colon X \to X$ is compact if and only if $\dim X < \infty$.

Hint: You can use point (b).

Dr. Silvio Fanzon
silvio.fanzon@uni-graz.at

Exercise 2.2 (20 pts) - Image through compact operators. Let X, Y be Banach spaces and assume that X is reflexive. Let $K \in \mathcal{L}(X,Y)$ and $M \subset X$ be closed, convex and bounded.

(a) Show that $K(M)$ is closed in Y.

Hint: Recall that a linear operator $K \colon X \to Y$ between normed spaces is continuous if and only if it is weak-to-weak continuous. Also recall that if $M \subset X$ is convex then the closure of M coincides with its weak closure.

(b) In addition, assume that $K \in \mathcal{K}(X,Y)$. Show that $K(M)$ is compact.

Exercise 2.3 (20 pts) - Compact operators on Hilbert spaces. Let X, Y be real Hilbert spaces and $K \in \mathcal{L}(X,Y)$. Let $x_n, x \in X$ for $n \in \mathbb{N}$.

a) Show that $x_n \to x$ strongly in X if and only if

$$x_n \rightharpoonup x \quad \text{weakly in} \quad X \quad \text{and} \quad \|x_n\|_X \to \|x\|_X \,.$$

b) Show that K is compact if and only if the following condition holds:

$$\text{If} \quad x_n \rightharpoonup x \quad \text{weakly in} \quad X\,, \quad \text{then} \quad Kx_n \to Kx \quad \text{strongly in} \quad Y\,.$$

Hint: The statements in Exercise 2.2 could be useful.

Exercise 2.4 (10 pts) - Range of compact operators.

a) Let X be a Banach space with $\dim X = +\infty$ and let $K \in \mathcal{K}(X)$. Show that K cannot be surjective, that is, there exists $y \in Y$ such that the equation

$$Kx = y$$

has no solution in X.

b) Let X, Y be Banach spaces. Let $K \in \mathcal{K}(X,Y)$. Show that $\mathrm{rg}(K)$ is closed if and only if $\dim \mathrm{rg}(K) < \infty$.

Hint: You might find Exercise 2.1 point (c) and Exercise 1.3 useful. Also recall that the composition (in whichever order) of a compact operator with a bounded operator is compact.

Exercise 2.5 (20 pts) Define the operator $K \colon \ell_2 \to \ell_2$ by

$$(Kx)_j := \frac{x_j}{j}\,, \qquad \text{for all } \; j \in \mathbb{N}\,.$$

(a) Show that K is well-defined, linear and compact.

Hint: ℓ_2 is a Hilbert space, therefore you can use Exercise 2.3 to prove compactness. Also, you could use the dominated convergence Theorem in the ℓ_1 setting.

(b) Show that range of K is not closed by finding some element $y \in \overline{\mathrm{rg}(K)} \smallsetminus \mathrm{rg}(K)$.