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Inverse Problems - Exercise Sheet 1

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Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $K : X \to Y$ be a linear continuous operator. For a given datum $f \in Y$ consider the inverse problem of finding $u \in X$ such that

$$Ku = f\,. \tag{1}$$

We say that the inverse problem (1) is *well-posed* in the sense of Hadamard if it admits unique solution for all $y \in Y$ and if solutions are continuous with respect to perturbations, i.e., if it holds

$$\|Ku_j - f\|_Y \to 0 \quad \Longrightarrow \quad \|u_j - u\|_X \to 0\,.$$

Problem (1) is *ill-posed* if it is not well-posed.

Exercise 1.1 - Matrix inversion (30 pts)

Let $n \geq 1$ and consider $X = Y = \mathbb{R}^n$. Suppose that $K \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix. Then by the spectral theorem

$$K = \sum_{j=1}^{n} \lambda_j \, k_j \otimes k_j$$

with $\{k_j\}_{j=1}^n$ orthonormal basis of $\mathbb{R}^n$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$ eigenvectors, where the tensor product between two vectors $a, b \in \mathbb{R}^n$ is the matrix $a \otimes b := ab^T$, that is, $(a \otimes b)_{ij} = a_i b_j$.

(a) Suppose that $Ku = f$ and $Ku^\delta = f^\delta$. Show that

$$\left\|u - u^\delta\right\| \leq \frac{1}{\lambda_n} \left\|f - f^\delta\right\| ,$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$.

Hint: Note that $(a \otimes b)u = (b \cdot u)a$, where $\cdot$ is the scalar product in $\mathbb{R}^n$.

(b) In the setting of (a), prove the relative error bound

$$\frac{\left\|u - u^\delta\right\|}{\|u\|} \leq \frac{\lambda_1}{\lambda_n} \frac{\left\|f - f^\delta\right\|}{\|f\|}\,.$$

(c) Is the inverse problem $Ku = f$ well-posed?

(d) $\kappa := \lambda_1 \lambda_n^{-1}$ is called *condition number* of the matrix K. In terms of real-world reconstructions, i.e., in the presence of noise, is it better for κ to be small or large?

Exercise 1.2 - Differentiation (30 pts)

Let $X = C([0,1])$ be the space of continuous functions equipped with the supremum norm

$$\|u\|_{\infty} = \sup_{x \in [0,1]} |u(x)| , \quad \text{for all} \;\; u \in X .$$

Let $Y = \{f \in C^1([0,1]) : \; f(0) = 0\}$, with $C^1([0,1])$ the space of continuously differentiable functions. Define the linear operator $K \colon X \to Y$ by

$$(Ku)(x) = \int_0^x u(y)\, dy ,$$

for all $u \in X$ and $x \in [0,1]$.

(a) Show that the inverse problem $Ku = f$ admits a unique solution for all $f \in Y$.

(b) Prove that the inverse problem $Ku = f$ is ill-posed when Y is equipped with the supremum norm.

Hint: Consider noisy data of the form $f^\delta = f + n^\delta$ for some suitable noise $n^\delta \in Y$.

(c) Show that the inverse problem $Ku = f$ is well-posed when Y is equipped with the norm $\|\cdot\|_{C^1}$, where $\|u\|_{C^1} = \|u\|_\infty + \|u'\|_\infty$ for all $u \in C^1([0,1])$.

Exercise 1.3 - Closed range (20 pts)

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and $K : X \to Y$ be a bounded linear operator. On X we define the equivalence relation

$$x_1 \sim x_2 \iff x_1 - x_2 \in \ker(K) .$$

Define $\hat{X} = X/_\sim$ to be the quotient space w.r.t. $\sim$, with equivalence classes denoted by $[x]$. Introduce

$$\|[x]\|_{\hat{X}} := \inf_{y \in [x]} \|y\|_X ,$$

and the operator $\hat{K} \colon \hat{X} \to \operatorname{rg}(K)$ defined by $\hat{K}[x] := Kx$, where $\operatorname{rg}(K)$ is the range of K.

(a) Show that $(\hat{X}, \|\cdot\|_{\hat{X}})$ is a Banach space.

Hint: Use that a normed space is complete if and only if every absolutely convergent series is convergent

(b) Show that $\hat{K}$ is well-defined, linear and bounded.

(c) Show that if $\operatorname{rg}(K)$ is closed then $\hat{K}^{-1}$ is continuous.

Hint: Note that $\hat{K}$ is bijective and use the open mapping theorem

(d) Prove that if $\hat{K}^{-1}$ is continuous then $\operatorname{rg}(K)$ is closed.

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On the space $L^2([-\pi,\pi]^2)$ we define the scalar product

$$\langle u,\, v\rangle := \int_{[-\pi,\pi]^2} u(x)\overline{v}(x)\,dx\,.$$

With respect to such scalar product, an orthonormal basis of $L^2([-\pi,\pi]^2)$ is given by the functions

$$e_l(x_1,x_2) := c_l\, e^{i(l_1x_1+l_2x_2)}\,,$$

for all $l=(l_1,l_2)\in\mathbb{Z}^2$, with $c_l\in\mathbb{R}$ suitable normalization constant. Any function $u\in L^2([-\pi,\pi]^2)$ admits a representation in terms of its Fourier series

$$u=\sum_{l\in\mathbb{Z}^2}\hat{u}_l\, e_l\,,\qquad \hat{u}_l := \langle u,\, e_l\rangle\,.$$

Recall that $\hat{u} := (\hat{u}_l)_l \in \ell^2(\mathbb{Z}^2)$. Also, an operator is compact if it is the limit of finite-range operators in operator norm. Recall that a space is finite dimensional if and only if its closed unit ball is compact.

Exercise 1.4 - Convolution (20 pts)

For $k\in L^2([-\pi,\pi]^2)$ define the convolution operator $K\colon L^2([-\pi,\pi]^2)\to L^2([-\pi,\pi]^2)$ by setting

$$Ku := k*u\,,\quad (k*u)(x) := \int_{[-\pi,\pi]^2} k(x-y)u(y)\,dy\,,$$

where we implicitly assume that k and u are extended periodically to the whole $\mathbb{R}^2$.

(a) Show that

$$\widehat{(Ku)}_l = \frac{1}{c_l}\,\hat{k}_l\hat{u}_l \quad\text{for all}\quad l\in\mathbb{Z}^2\,.$$

Moreover, provide the inverse of K in case $\hat{k}_l\neq 0$ for all $l\in\mathbb{Z}^2$.

(b) Prove that K is compact. Deduce that, in case $\hat{k}_l\neq 0$ for infinitely many $l\in\mathbb{Z}^2$, $\operatorname{rg}(K)$ is not closed.

Note: Thanks to Exercise 1.3, this shows that de-convolution is an ill-posed inverse problem