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Analysis 3 - Exercise Sheet 13

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Theoretical preliminaries: Let $G \subset \mathbb{R}^d$ a bounded, open set and $F : \tilde{G} \to \mathbb{R}^d$, $f : \tilde{G} \to \mathbb{R}$ continuously differentiable with $\tilde{G}$ open and $\overline{G} \subset \tilde{G}$. We denote

$$\mathrm{div} F = \sum_{i=1}^{d} \partial_{x_i} F_i,$$

$$\Delta f = \sum_{i=1}^{d} \partial_{x_i}^2 f.$$

Further $\nu(x)$ denotes the unit normal vector on ∂G in $x \in \partial G$ which points outside of G. We denote the surface measure for integration on the boundary of G as σ.
Gauss' theorem It holds true that

$$\int_G \mathrm{div}(F)\, d(x,y) = \int_{\partial G} \langle F, \nu \rangle \, d\sigma.$$

Exercise 12.1 (20 pts) Show that

$$\int_{\mathbb{R}} e^{-z^2}\, dz = \sqrt{\pi}$$

by computing

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, d(x,y).$$

Hint: Polar coordinates.

Exercise 12.2 (20 pts) Compute the surface volume of a general torus with $0 < r < R$ by integrating the function $f \equiv 1$ over the surface.

Exercise 12.3 (20 pts) Let $G \subset \mathbb{R}^d$ a bounded, open set and $F : \tilde{G} \to \mathbb{R}^d$, $f, g : \tilde{G} \to \mathbb{R}$, f twice and F, g once continuously differentiable with $\tilde{G}$ open and $\overline{G} \subset \tilde{G}$. Use Gauss' theorem to prove that

a)

$$\int_G \langle \nabla f, F \rangle \, dx = \int_{\partial G} \langle fF, \nu \rangle \, d\sigma - \int_G f \mathrm{div} F \, dx.$$

b)

$$\int_G -\Delta f g = \int_G \langle \nabla f, \nabla g \rangle - \int_{\partial G} g \langle \nabla f, \nu \rangle \, \sigma.$$

We prove a version of Gauss' theorem Obviously, do not use Gauss' theorem in the following.
Let $a < 0$, $b > 1$ and $f : (a,b) \to (0,\infty)$ be a strictly increasing diffeomorphism. Denote the set $G = \{(x,y) \mid x \in (0,1),\ 0 < y < f(x)\}$. Let $F = (F_1, F_2)$ be a continuously differentiable vector field defined in a neighborhood of $\overline{G}$. We will show together that

$$\int_G \mathrm{div}(F)\, d(x,y) = \int_{\partial G} \langle F, n \rangle \, ds.$$

Exercise 12.4 (20 pts) Denoting $\alpha = f(0)$, $\beta = f(1)$, use the fundamental theorem of calculus to show that

$$\int_G \operatorname{div} F \; d(x,y) = \int_0^1 -F_2(x,0)\; dx + \int_0^1 F_2(x, f(x))\; dx + \int_0^\beta F_1(1,y)\; dy + \int_0^\alpha -F_1(0,y)\; dy + \int_0^1 -F_1(x,f(x)) f'(x)\; dx \tag{1}$$

Exercise 12.5 (20 pts) Compute the line integral $\int_{\partial G} \langle \nabla F, n\rangle \; ds$ by dividing it into 4 segments $\{(x,y) \in \delta G \mid x = 0,\; y \in (0,\alpha)\}$, $\{(x,y) \in \delta G \mid x = 1,\; y \in (0,\beta)\}$, $\{(x,y) \in \delta G \mid x \in (0,1),\; y = 0\}$, $\{(x,y) \in \delta G \mid x \in (0,1),\; y = f(x)\}$ to show that it agrees with the above from 12.4. Why can we neglect the corners $(x,y) = (0,0)$, $(x,y) = (1,0)$, etc.