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Analysis 3 - Exercise Sheet 7

Publication date: November 23, 2022

Due date: November 30, 2022

Note. We assume the definitions given in Exercise Sheet 6. Also, all the statements in Exercise Sheet 6 can be used to solve the problems below.

Exercise 7.1 (20 pts)

(a) Suppose that $f\colon [0,1] \to [0,\infty)$ is continuous and such that $f(1) = 0$. Show that there exists $\hat{x} \in [0,1]$ such that $f(\hat{x}) = \hat{x}$.

Hint: $A \subset \mathbb{R}$ is connected if and only if A is an interval.

(b) Let $n \geq 1$. Suppose that $f\colon \mathbb{S}^n \to \mathbb{R}$ is continuous. Show that there exists $\hat{x} \in \mathbb{S}^n$ such that $f(\hat{x}) = f(-\hat{x})$.

Hint: Recall that $\mathbb{S}^n$ is connected for all $n \geq 1$. Also it might be useful to consider $g(x) := f(x) - f(-x)$.

Definition. Consider the points $(z_1, \dots, z_m)$ with $z_i \in \mathbb{R}^n$. Denote by S_k the line segment connecting z_k to z_{k+1}, that is, $S_k := [z_k, z_{k+1}]$. The set $P = \cup_{i=1}^{m-1} S_i$ is called *polygonal path* through $(z_1, \dots, z_m)$. We also say that P connects z_1 to z_m.

Definition. A subset $A \subset \mathbb{R}^n$ is called *polygonally path-connected* if for every $x, y \in A$ there exists a polygonal path $P \subset A$ connecting x to y.

Exercise 7.2 (20 pts) Fix some integer $n \geq 2$ and let $A \subset \mathbb{R}^n$ be countable. Prove that $\mathbb{R}^n \setminus A$ is polygonally path-connected.

Exercise 7.3 (20 pts) Fix some integer $n \geq 2$ and Let $A \subset \mathbb{R}^n$ be convex and bounded. Prove that $\mathbb{R}^n \setminus A$ is path-connected.

Exercise 7.4 (20 pts) Let $A \subset \mathbb{R}^n$ be open. Prove that A is connected if and only if it is polygonally path-connected.

Let $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{R}^n$ with $\gamma = (\gamma_1, \dots, \gamma_n)$ be a curve.

Definition. We say that γ is *regular* if $\gamma_i \in C^1([a,b])$ for all $i = 1, \dots, n$ and

$$\|\gamma'(t)\| = \sqrt{(\gamma_1'(t))^2 + \dots + (\gamma_n'(t))^2} > 0$$

for all $t \in (a,b)$. We say that γ is *piecewise regular* if there exist $a_1 = a < a_2 < \ldots < a_k = b$ such that γ is regular in each $[a_i, a_{i+1}]$.

Theorem. Let $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{R}^n$ be piecewise regular. Then the length of γ is given by

$$\ell(\gamma) = \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} \sqrt{(\gamma_1')^2 + \dots + (\gamma_n')^2}\, dt\,.$$

Definition. Let $\gamma\colon [a,b] \subset \mathbb{R} \to \mathbb{R}^n$ be piecewise regular and $F\colon \gamma([a,b]) \subset \mathbb{R}^n \to \mathbb{R}$ continuous. The integral of F along γ is defined by

$$\int_\gamma F\, ds := \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} F(\gamma(t)) \sqrt{(\gamma_1')^2 + \dots + (\gamma_n')^2}\, dt\,.$$

Exercise 7.5 (20 pts) Let $\gamma\colon [0,\pi] \to \mathbb{R}^2$ defined by

$$\gamma(t) := (\cos^3 t, \sin^3 t)\,.$$

(a) Prove that γ is piecewise regular.

(b) Compute $\ell(\gamma)$.

(c) Compute $\int_\gamma F\, ds$ where $F\colon \mathbb{R}^2 \to \mathbb{R}$ is defined by $F(x,y) := \sqrt[3]{|xy|}$.