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Analysis 3 - Exercise Sheet 6

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Exercise 6.1 (20 pts) Define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x,y) := \left(\sin(xy) + x\cos y, e^{x+y} - \frac{1}{1+x^2+y^2}\,. \right)$$

(a) Show that F is locally invertible in $(0,0)$.

(b) Denote by $G = (G^1, G^2)$ the local inverse of F around $(0,0)$. Compute the first order Taylor approximation of G^i around $(0,0)$, that is,

$$G^i(x,y) = G^i(0,0) + G^i_x(0,0)x + G^i_y(0,0)y + o(\sqrt{x^2+y^2})\,,$$

for $i = 1,2$.

In the following we assume familiarity with the concepts of *topology*, *induced topology*, *continuity* for maps between topological spaces, and *homeomorphism* of topological spaces.

Definition. A topological space X is called *connected* if the only subsets that are both open and closed are $\emptyset$ and X. A non-connected topological space is called *disconnected*.

Exercise 6.2 (20 pts) Let X be a topological space. Prove that the following statements are equivalent:

(a) X is disconnected.

(b) $X = A_1 \cup A_2$ with A_1, A_2 open, disjoint and proper, i.e., $A_i \neq \emptyset$, $A_i \neq X$.

(c) $X = C_1 \cup C_2$ with C_1, C_2 closed, disjoint and proper, i.e., $C_i \neq \emptyset$, $C_i \neq X$.

Definition. The euclidean topology on $\mathbb{R}^n$ is defined as the collection of sets

$$\tau := \{A \subset \mathbb{R}^n : \forall\, x \in A,\, \exists\, \varepsilon > 0 \text{ such that } B_\varepsilon(x) \subset A\}\,,$$

where $B_\varepsilon(x) := \{y \in \mathbb{R}^n : \ \|y - x\| < \varepsilon\}$ and $\|\cdot\|$ is the euclidean norm.

Note. In the following $\mathbb{R}^n$ is always equipped with the euclidean topology. Any subset $A \subset \mathbb{R}^n$ is itself a topological space with the topology induced by the euclidean one.

Definition. A topological space X is called *path-connected* if for every pair $x, y \in X$ there exists a continuous map $\alpha\colon [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. The map α is a *path* between x and y.

Note. In the following you can assume this fact: $I \subset \mathbb{R}$ is connected if and only if it is an interval.

Exercise 6.3 (20 pts) Let X and Y be topological spaces.

(a) Assume that $f\colon X \to Y$ is continuous. Show that if X is connected then $f(X)$ is connected.

(b) Show that if X is path-connected then it is connected.

(c) Suppose that $A, B \subset X$ are path-connected and such that $A \cap B \neq \emptyset$. Show that $A \cup B$ is path-connected.

(d) Assume that $f\colon X \to Y$ is continuous. Show that if X is path-connected then $f(X)$ is path-connected.

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Exercise 6.4 (20 pts)

(a) Define $X := \{x \in \mathbb{R} : \ x \neq 0\}$. Prove that X is disconnected.

(b) Let $a, b \in \mathbb{R}^n$ and define the line segment

$$[a, b] := \{ta + (1-t)b \, , \, t \in [0, 1]\} \, .$$

Prove that $[a, b]$ is path-connected.

(c) Let $C \subset \mathbb{R}^n$ be convex. Show that C is path-connected.

Exercise 6.5 (20 pts)

(a) Let $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \ \|x\| = 1\}$. Show that $\mathbb{S}^n$ is path-connected.

(b) Show that $\mathbb{S}^1$ is not homeomorphic to $(0, 1)$.

(c) Prove that the intervals $(0, 1)$ and $(0, 1]$ are not homeomorphic.