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Analysis 3 - Exercise Sheet 5

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Exercise 5.1 (25 pts) Let $g\colon \mathbb{R} \to \mathbb{R}$ be differentiable in $t = 0$. Moreover suppose g is bounded, that is, there exists $M \geq 0$ such that $|g(t)| \leq M$ for all $t \in \mathbb{R}$. Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) := \begin{cases} x^2\, g\left(\dfrac{y}{x}\right) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Prove that $F_{xy}(0,0) = F_{yx}(0,0)$ if and only if $g'(0) = 0$.

Define $\mathbb{S}^{n-1} := \{v \in \mathbb{R}^n : \ \|v\| = 1\}$.

Theorem 1. Let $A \subset \mathbb{R}^n$ be open, $F\colon A \to \mathbb{R}$. If F is differentiable in $z_0 \in A$, then F admits all the directional derivatives in z_0 and

$$F_v(z_0) = \nabla F(z_0) \cdot v = \sum_{i=1}^{n} F_{x_i}(z_0) v_i\,, \quad \text{for all } v \in \mathbb{S}^{n-1}\,. \tag{1}$$

The next exercise shows that, in general, formula (1) does not hold if F is not differentiable.

Exercise 5.2 (25 pts) Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) := \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

(a) Prove that $F_v(0,0)$ exists for all $v \in \mathbb{S}^1$ and compute it.

(b) Prove that (1) does not hold, i.e., that there exists some $v \in \mathbb{S}^1$ such that

$$F_v(0,0) \neq \nabla F(0,0) \cdot v\,.$$

(c) Can F be differentiable in $(0,0)$?

Definition. Consider a vector valued function $F = (F^1, \dots, F^n)\colon A \subset \mathbb{R}^n \to \mathbb{R}^n$. The Jacobian of F at $z \in A$ is defined as the $n \times n$ matrix of partial derivatives

$$J_F(z) := \left(F^i_{x_j}(z)\right)_{ij}\,.$$

Inverse Function Theorem. Let $A \subset \mathbb{R}^n$ be open. Let $F\colon A \to \mathbb{R}^n$ be a C^1 function and suppose that

$$\det J_F(z_0) \neq 0$$

for some $z_0 \in A$. Then F is *locally invertible* around z_0, that is, there exist $U \subset A$ neighbourhood of z_0, V neighbourhood of $F(z_0)$ and a C^1 function $G\colon V \to U$ such that $(F \circ G)(w) = w$ for all $w \in V$ and $(G \circ F)(z) = z$ for all $z \in U$. We denote $F^{-1} := G$. In particular for all $w \in V$ it holds

$$J_{F^{-1}}(w) = \left[J_F(F^{-1}(w))\right]^{-1}\,.$$

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Exercise 5.3 (25 pts)

(a) Consider the map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F(x, y, z) = (xz, 2xy, 3yz)\,.$$

For which points of $\mathbb{R}^3$ is the map F locally invertible?

(b) Consider the map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x, y) = (e^x \cos y, e^x \sin y)\,.$$

Show that F is locally invertible for every point in $\mathbb{R}^2$. Is F globally invertible?

Exercise 5.4 (25 pts) Suppose $F \in C^2(\mathbb{R}^2)$ and that there exists $(x_0, y_0) \in \mathbb{R}^2$ such that

$$F(x_0, y_0) = F_x(x_0, y_0) = F_y(x_0, y_0) = 0\,.$$

Moreover assume that

$$F_{xx}(x_0, y_0)F_{yy}(x_0, y_0) > F_{xy}^2(x_0, y_0)\,.$$

Use the Inverse Function Theorem and the Minimality/Maximality Criterion from Exercise Sheet 2 to prove the existence of a neighbourhood U of (x_0, y_0) such that

$$F(x, y) \neq 0 \quad \text{for all} \;\; (x, y) \in U \setminus \{(x_0, y_0)\}\,.$$