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Analysis 3 - Exercise Sheet 4

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Consider the following statement you saw in class.

Theorem 1. Let $A \subset \mathbb{R}^n$ be open and $F\colon A \to \mathbb{R}$. Suppose that there exist $z_0 \in A$ and a neighbourhood $U \subset A$ of z_0 such that ∇F exists and is continuous in U. Then F is differentiable in z_0.

The converse of Theorem 1 does not hold, as shown in the next exercise.

Exercise 4.1 (25 pts) Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) := \begin{cases} 0 & \text{if } y = 0\,, \\ y^2 \cos\left(\dfrac{1}{y}\right) & \text{if } y \neq 0\,. \end{cases}$$

(a) Compute F_x and F_y. Prove that F_y is not continuous in $(x,0)$ for all $x \in \mathbb{R}$.

(b) Prove that F is differentiable in $(x,0)$ for all $x \in \mathbb{R}$.

In class you saw the following theorem.

Theorem 2. Let $A \subset \mathbb{R}^n$ be open and $F\colon A \to \mathbb{R}$. Suppose that there exist $z_0 \in A$ and a neighbourhood $U \subset A$ of z_0 such that $\nabla^2 F$ exists and is continuous in U. Then $F_{x_i x_j}(z_0) = F_{x_j x_i}(z_0)$ for all i,j in $\{1,\dots,n\}$.

The aim of the next exercise is to prove that the assumption of $\nabla^2 F$ being continuous cannot be removed.

Exercise 4.2 (25 pts) Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) := \begin{cases} 0 & \text{if } (x,y) = (0,0)\,, \\ \dfrac{x^3 y}{x^2 + y^2} & \text{if } (x,y) \neq (0,0)\,. \end{cases}$$

(a) Prove that F is continuous in $\mathbb{R}^2$.

(b) Compute $\nabla F = (F_x, F_y)$ and prove that F is differentiable in $\mathbb{R}^2$.

(c) Prove that F_{xy} and F_{yx} exist in $\mathbb{R}^2$ and that

$$F_{xy}(0,0) \neq F_{yx}(0,0)\,.$$

(d) Check that F_{xy} and F_{yx} are not continuous in $(0,0)$.

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For the next exercise it will be useful to recall the following Taylor formula in one dimension.

Theorem 3. Let $a, b \in \mathbb{R}$ and $g \in C^2(I)$ with $I = [a,b] \subset \mathbb{R}$. Let $t \in I$ and $s > 0$ be such that $t + s \in I$. Then there exists $\xi \in (0,1)$ such that

$$g(t+s) = g(t) + g'(t)s + \frac{1}{2} g''(t + \xi s) s^2 \,.$$

Exercise 4.3 (25 pts) Let $A \subset \mathbb{R}^2$ be open and $F \in C^2(A)$.

(a) Let $(x,y), (h,k) \in \mathbb{R}^2$ be such that $P_t := (x + th, y + tk) \in A$ for all $t \in [0,1]$. Using Theorem 3, show that there exists $\xi \in (0,1)$ such that

$$F(x+h, y+k) = F(x,y) + F_x(x,y)h + F_y(x,y)k + \frac{1}{2}\left\{ F_{xx}(P_\xi)h^2 + 2F_{xy}(P_\xi)hk + F_{yy}(P_\xi)k^2 \right\} \,.$$

(b) The second order Taylor polynomial of F in $(0,0)$ is defined by

$$P_2(x,y) := F(0,0) + F_x(0,0)x + F_y(0,0)y + \frac{1}{2}\left\{ F_{xx}(0,0)x^2 + 2F_{xy}(0,0)xy + F_{yy}(0,0)y^2 \right\} \,.$$

Compute P_2 for $F \colon \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x,y) := (2x + y)e^{x^2 - y^2}$.

Defintion. Define $\mathbb{S}^n := \{ v \in \mathbb{R}^{n+1} : \; \|v\| = 1 \}$. Let $A \subset \mathbb{R}^{n+1}$ be open, $F \colon A \to \mathbb{R}$, and $v \in \mathbb{S}^n$. The directional derivative of F at $z_0 \in A$ in direction v is defined by

$$F_v(z_0) := \lim_{t \to 0} \frac{F(z_0 + tv) - F(z_0)}{t} \,,$$

whenever the limit exists.

Theorem 4. Let $A \subset \mathbb{R}^{n+1}$ be open, $F \colon A \to \mathbb{R}$. If F is differentiable in $z_0 \in A$, then F admits all the directional derivatives in z_0 and

$$F_v(z_0) = \nabla F(z_0) \cdot v = \sum_{i=1}^{n+1} F_{x_i}(z_0) v_i \,, \qquad \text{for all } \; v \in \mathbb{S}^n \,.$$

The next exercise shows that the converse of Theorem 4 does not hold, i.e., there exists F which admits all the directional derivatives at some point z_0, but is not differentiable at z_0.

Exercise 4.4 (25 pts) Define $F \colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) := \begin{cases} 0 & \text{if } \; (x,y) = (0,0)\,, \\ \dfrac{x^3 y}{x^4 + y^2} + y & \text{if } \; (x,y) \neq (0,0)\,. \end{cases}$$

(a) Prove that $F_v(0,0)$ exists for all $v \in \mathbb{S}^1$ and compute it.

(b) Prove that F is not differentiable in $(0,0)$.

(c) Prove that for all $v \in \mathbb{S}^1$ it holds

$$F_v(0,0) = \nabla F(0,0) \cdot v \,.$$