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Analysis 3 - Exercise Sheet 3

Publication date: October 12, 2022 Due date: October 19, 2022

Remark: The following exercise addresses a question that a few of you asked in the previous class. Enjoy!

Exercise 3.1 (25 pts)

(a) Suppose that $z = 0$ is a local minimizer for a given function $F\colon \mathbb{R}^n \to \mathbb{R}$. Let $v \in \mathbb{R}^n \setminus \{0\}$ and consider the restriction of F along the line of direction v, that is, the function $g_v(t) := F(tv)$ for $t \in \mathbb{R}$. Prove that $t = 0$ is a local minimizer for g_v.

(b) We now show that the converse of point (a) does not hold, even if F is smooth. To this end, consider the function $F\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x, y) := (y - x^2)(y - 2x^2)\,.$$

Prove the following statements:

(i) $(0, 0)$ is the only critical point of F.

(ii) The Hessian of F vanishes in $(0, 0)$.

(iii) Consider the restriction of F along the lines through the origin:

$$g_m(x) := \begin{cases} F(x, mx) & \text{if } \ m \in \mathbb{R}\,, \\ F(0, x) & \text{if } \ m = \infty\,. \end{cases}$$

Show that for all $m \in \mathbb{R} \cup \{\infty\}$ the point $x = 0$ is a local minimizer for g_m.

(iv) Show that $(0, 0)$ is a saddle point for F.

Hint: To understand what is happening, it might be helpful to draw the sets in $\mathbb{R}^2$ where F is positive, negative and zero.

Implicit Function: Let $A \subset \mathbb{R}^2$ be open, and $F\colon A \to \mathbb{R}$. Define the set

$$Z := \{(x, y) \in A : \ F(x, y) = 0\}\,.$$

Assume $(x_0, y_0) \in Z$, i.e., $F(x_0, y_0) = 0$. We say that the equation $F = 0$ defines an implicit function $y = f(x)$ at the point (x_0, y_0) if the set Z coincides with the graph of f around (x_0, y_0), that is, if there exist $\varepsilon, \delta > 0$ and $f\colon I_\varepsilon(x_0) \to I_\delta(y_0)$, with $I_\varepsilon(x_0) := [x_0 - \varepsilon, x_0 + \varepsilon], I_\delta(y_0) := [y_0 - \delta, y_0 + \delta]$ such that

$$\{(x, y) \in I_\varepsilon(x_0) \times I_\delta(y_0) : \ F(x, y) = 0\} = \{(x, f(x)) : \ x \in I_\varepsilon(x_0)\}\,.$$

Clearly, we say that $F = 0$ defines an implicit function $x = f(y)$ at the point (x_0, y_0) if the set Z coincides with the graph of f around (x_0, y_0)

Implicit Function Theorem: Let $A \subset \mathbb{R}^2$ be open, and $F \colon A \to \mathbb{R}$ with $F \in C^1(A)$. Assume there exists a point $(x_0, y_0) \in A$ such that

$$F(x_0, y_0) = 0\,, \quad F_y(x_0, y_0) \neq 0\,.$$

Then there exist $\varepsilon, \delta > 0$ and a unique implicit function $f \colon I_\varepsilon(x_0) \to I_\delta(y_0)$, i.e., f satisfies $f(x_0) = y_0$ and

$$F(x, f(x)) = 0\,, \quad \text{for all } x \in I_\varepsilon(x_0)\,.$$

Moreover $f \in C^1(I_\varepsilon(x_0), I_\delta(y_0))$ and

$$f'(x) = -\frac{F_x(x, f(x))}{F_y(x, f(x))}\,.$$

Clearly, the analogous statement holds if

$$F(x_0, y_0) = 0\,, \quad F_x(x_0, y_0) \neq 0\,,$$

and in this case $F = 0$ defines an implicit function $x = f(y)$.

Tangent line to a set: Let $A \subset \mathbb{R}^2$ be open, and $F \colon A \to \mathbb{R}$ with $F \in C^1(A)$. Define the set

$$Z := \{(x, y) \in A : \; F(x, y) = 0\}\,.$$

Suppose that the point $(x_0, y_0) \in Z$ is such that either $F_x(x_0, y_0) \neq 0$ or $F_y(x_0, y_0) \neq 0$. Then the equation of the tangent line to Z at (x_0, y_0) is given by

$$F_x(x_0, y_0)\,(x - x_0) + F_y(x_0, y_0)\,(y - y_0) = 0\,.$$

Exercise 3.2 (25 pts) Let $F \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = x^3 + y^3 - 3xy\,.$$

Find the points $(x_0, y_0) \in \mathbb{R}^2$ such that $F = 0$ defines implicitly a map $y = f(x)$.

Exercise 3.3 (25 pts) Let $F \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = 2y^3 + 4x^2 y - 3x^4 + x + 6y\,.$$

Prove that the equation $F = 0$ defines an implicit function $y = f(x)$ for all $(x_0, y_0) \in \mathbb{R}^2$.

Exercise 3.4 (25 pts) Let $F \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = x^3 + y^3 - 4x^2 y + 2\,.$$

(a) Show that the equation $F = 0$ defines an implicit function $y = f(x)$ around the point $(1, 1)$.

(b) Compute $f'(1)$.

(c) Compute the equation of the line tangent to the set

$$Z = \{(x, y) \in \mathbb{R}^2 : \; F(x, y) = 0\}$$

at the point $(1, 1)$.