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Analysis 3 - Exercise Sheet 2

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Recall: Let $A \subset \mathbb{R}^n$ be an open set. Suppose $f: A \to \mathbb{R}$ is differentiable at $z^* \in \mathbb{R}^n$. We say that z^* is a critical point of f if $\nabla f(z^*) = 0$. Recall that a local minimizer or maximinizer of f is always a critical point. Suppose now $n = 2$ and that f is C^2. The Hessian of f is defined by

$$Hf(x,y) := \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx} f_{yy} - (f_{xy})^2 \,.$$

Suppose (x^*, y^*) is a critical point of f. If

$$Hf(x^*, y^*) > 0 \,, \quad f_{xx}(x^*, y^*) > 0 \,,$$

then (x^*, y^*) is a local mimizer for f. If

$$Hf(x^*, y^*) > 0 \,, \quad f_{xx}(x^*, y^*) < 0 \,,$$

then (x^*, y^*) is a local maximizer for f. If

$$Hf(x^*, y^*) < 0 \,,$$

then (x^*, y^*) is a saddle point for f.

Exercise 2.1 (25 pts) Define $f: \mathbb{R}^2 \to \mathbb{R}$ by setting

$$f(x,y) := \frac{xy}{1 + x^2 + y^2} \,.$$

Find all the critical points of f and classify them into local maximizers, local minimizers and saddle points.

Exercise 2.2 (25 pts) Define $f: \mathbb{R}^2 \to \mathbb{R}$ by setting

$$f(x,y) := 2(x^4 + y^4 + 1) - (x+y)^2 \,.$$

Find all the critical points of f and classify them into local maximizers, local minimizers and saddle points.

Hint: if at some critical point (x^*, y^*) one has $Hf(x^*, y^*) = 0$, then nothing can be concluded about the nature of (x^*, y^*). In such case, one has to proceed manually, for example by considering the restriction of f to the line through the origin $\{y = mx\}$ for $m \in \mathbb{R}$, and analyze the resulting function of one variable.

Exercise 2.3 (25 pts) Define $f: \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = xy^2$ and consider the set

$$A = \{(x,y) \in \mathbb{R}^2 : \; y \geq 0, \; y \leq 1 + x, \; y \leq 1 - x\} \,.$$

Find the global maximizers and minimizers of f restricted to the set A.

Hint: Draw A and consider the separate cases of internal points and boundary points.

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Exercise 2.4 (25 pts)

(a) Suppose $f\colon A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x^* \in \mathring{A}$, with $\mathring{A}$ denoting the interior of A. Show that if x^* is a local minimizer or maximizer for f, then $\nabla f(x^*) = 0$.

(b) (Rolle's Theorem in $\mathbb{R}^n$) Suppose that $A \subset \mathbb{R}^n$ is compact with $\mathring{A} \neq \emptyset$. Let $f\colon A \to \mathbb{R}$ be continuous in A, differentiable in $\mathring{A}$, and constant on ∂A. Prove that there exists $x^* \in \mathring{A}$ such that $\nabla f(x^*) = 0$.

Hint: Use Weierstrass' Theorem.