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Analysis 3 - Exercise Sheet 1

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We start with some revision exercises on Analysis 2 topics

Exercise 1.1 (20 pts) Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) := f(x+2y) + f(7y-3x)\,,$$

for all $x, y \in \mathbb{R}$. Is F differentiable? In that case, compute ∇F.

Exercise 1.2 (20 pts) Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by setting $F(x,y) := \sqrt{|xy|}$. Is F differentiable in $(0,0)$? Justify your answer.

Recall: Let (X,d) be a non-empty complete metric space and $F\colon X \to X$. We say that F is a *contraction* if there exists a constant $C \in [0,1)$ such that

$$d(F(z_1), F(z_2)) \leq C d(z_1, z_2)$$

for all $z_1, z_2 \in X$. We say that z^* is a *fixed point* for F if $F(z^*) = z^*$. The Banach fixed point theorem states that if F is a contraction, then F admits a unique fixed point. Recall that $\mathbb{R}^n$ is a complete metric space with the Euclidean distance.

Exercise 1.3 (20 pts) Let (X,d) be a non-empty complete metric space. Prove the Banach fixed point theorem stated above.

Exercise 1.4 (20 pts) Define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$F(x,y) := (x + y/2, x/2 + y + 1)\,.$$

Define the map $G(x,y) = (x,y) - F(x,y)$. Using the Banach fixed point theorem on G, prove that F admits a unique zero, i.e., there exists a unique $(x^*, y^*) \in \mathbb{R}^2$ such that $F(x^*, y^*) = (0,0)$.

Exercise 1.5 (20 pts) Define $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x,y,z) := (x+y+z, xy+yz+zx, xyz)\,.$$

Determine all the points in $\mathbb{R}^3$ in which F is locally invertible.