• Γ-limsup ineq: Let $x \in X$. Since (*) holds, one can show that there there $\exists \{x_n\}$ s.t.

$$③ \quad x_n \to x \quad \text{and} \quad \liminf_{n\to+\infty} f_n(x_n) = \limsup_{n\to+\infty} f_n(x_n) = f(x)$$

concluding. $\square$

LESSON 15 - 23 JUNE 2021

FUNDAMENTAL THEOREM OF Γ-CONVERGENCE

We now want to show that the Γ-limit captures the asymptotic behavior of minimizers for a sequence $f_n : X \to \overline{\mathbb{R}}$.

LEMMA 11.8 (X,d) metric space, $f_n : X \to \overline{\mathbb{R}}$, $f_n \xrightarrow{\Gamma} f$. Then f is LSC.

Proof Assume $x_n \to x$. We need to show

$$f(x) \le \liminf_{n\to+\infty} f(x_n).$$

Since $f_k \xrightarrow{\Gamma} f$ we know that for each x_n there $\exists$ a recovery sequence $\{y_k\}$ s.t.

$$\lim_{k\to+\infty} y_k = x_n \quad , \quad f(x_n) = \lim_{k\to+\infty} f_k(y_k)$$

Therefore by a diagonal argument we can find $\{\tilde{y}_n\}$ s.t.

$$(*) \quad d(\tilde{y}_n, x_n) < \frac{1}{n} \quad , \quad |f_n(\tilde{y}_n) - f(x_n)| < \frac{1}{n} \quad , \quad \forall n \in \mathbb{N}.$$

Since $x_n \to x$, the first condition implies $\tilde{y}_n \to x$. Therefore,

Γ- liminf inequality as $\tilde{y}_n \to x$ — Second condition in (*) — As $1/n \to 0$

$$f(x) \le \liminf_{n\to+\infty} f_n(\tilde{y}_n) \le \liminf_{n\to+\infty} \left[f(x_n) + 1/n \right] = \liminf_{n\to+\infty} f(x_n)$$

concluding that f is LSC. $\square$

PROPOSITION 11.9

(X, d) metric space, $f_n : X \to \overline{\mathbb{R}}$, $f_n \xrightarrow{\Gamma} f$.

1. Let $A \subseteq X$ be open. Then

$$\limsup_{n\to+\infty} \left\{ \inf_{x\in A} f_n(x) \right\} \le \inf_{x\in A} f(x)$$

2. Let $K \subseteq X$ be compact. Then

$$\liminf_{n\to+\infty} \left\{ \inf_{x\in K} f_n(x) \right\} \ge \inf_{x\in K} f(x)$$

Proof 1. Fix $\varepsilon > 0$. By definition of inf there $\exists\ \hat{x} \in A$ s.t.

$$(*) \qquad f(\hat{x}) \le \inf_{x\in A} f(x) + \varepsilon$$

Let x_n be a recovery sequence for $\hat{x}$, i.e.

$$x_n \to \hat{x} \quad \text{and} \quad f(\hat{x}) \ge \limsup_{n\to+\infty} f_n(x_n)$$

Since A is open, $\hat{x} \in A$, and $x_n \to \hat{x}$, then $x_n \in A$ for $n >> 0$. Then

$$\inf_{x\in A} f(x) + \varepsilon \overset{(*)}{\geq} f(\hat{x}) \overset{x_n \text{ rec. seq. for } \hat{x}}{\geq} \limsup_{n\to+\infty} f_n(x_n) \overset{\text{As } x_n\in A \text{ for } n\gg 0}{\geq} \limsup_{n\to+\infty} \inf_{x\in A} f_n(x)$$

As ε is arbitrary, we conclude.

(2) Let $\{x_n\} \subseteq K$ be a sequence of quasi-minimizers, i.e.

$$(**) \qquad f_n(x_n) \leq \inf_{x\in K} f_n(x) + \frac{1}{n}, \qquad \forall n\in\mathbb{N}.$$

Since K is compact, up to extracting a subsequence, we can suppose

$$x_n \to \hat{x} \quad \text{for some} \quad \hat{x}\in K.$$

Then

$$\inf_{x\in K} f(x) \overset{\text{As } \hat{x}\in K}{\leq} f(\hat{x}) \overset{\Gamma\text{-liminf ineq., as } x_n\to\hat{x}}{\leq} \liminf_{n\to+\infty} f_n(x_n)$$

$$\overset{(**)}{\leq} \liminf_{n\to+\infty} \left[\inf_{x\in K} f_n(x) + \frac{1}{n}\right] \overset{\text{as } 1/n\to 0}{=} \liminf_{n\to+\infty} \left[\inf_{x\in K} f_n(x)\right]$$

$\square$

DEFINITION 11.10 (EQUICOERCIVITY)

(X,d) metric space, $f_n : X \to \mathbb{R}$. We say that $\{f_n\}$ is EQUICOERCIVE if $\exists\, K\subseteq X$ non empty and compact s.t.

$$\inf\{f_n(x) : x\in X\} = \inf\{f_n(x) : x\in K\}, \qquad \forall n\in\mathbb{N}.$$

(K is independent on n)

REMARK 11.11 (X,d) metric space, $f_n : X \to \mathbb{R}$. Suppose that there $\exists$ $M \in \mathbb{R}$ s.t. the set

$$\{ x \in X \mid f_n(x) \leq M, \ \forall n \in \mathbb{N} \}$$

is non empty and pre-compact. Then $\{f_n\}$ is EQUICOERCIVE.

Proof Set $K := \{ x \in X \mid f_n(x) \leq M, \ \forall n \in \mathbb{N} \}$. This is non empty, compact, and satisfies the condition $\inf_{x \in X} f_n = \inf_{x \in K} f_n$ for all $n \in \mathbb{N}$. $\square$

We are finally able to prove the main result of this section.

THEOREM 11.12 (CONVERGENCE OF MINIMUMS AND MINIMIZERS)

(X,d) metric space, $f_n : X \to \overline{\mathbb{R}}$. Suppose that:

(i) $\{f_n\}$ is equicoercive WRT the compact set K

(ii) $f_n \xrightarrow{\Gamma} f$ for some $f : X \to \overline{\mathbb{R}}$

Then: 1. f admits minimum on X

2. As $n \to +\infty$ we have $\inf_{x \in X} f_n(x) \to \min_{x \in X} f(x)$

3. Assume $\{x_n\}$ is a sequence of almost-minimizers, i.e.,

$$\lim_{n \to +\infty} \left\{ f_n(x_n) - \inf_{x \in X} f_n(x) \right\} = 0.$$

Suppose that $x_{n_k} \to \hat{x}$. Then $\hat{x}$ is minimum for f over X.

Proof ① By LEMMA 11.8 we know that the Γ-limit f is LSC.
Since K is compact, by the DIRECT METHOD (THM 9.4) there $\exists\ \hat{x} \in K$ s.t.

$$(K) \qquad f(\hat{x}) = \min_{x \in K} f(x) \qquad \text{(f admits minimum on K)}$$

We claim that

$$(*) \qquad f(\hat{x}) = \min_{x \in X} f(x) \qquad (\hat{x} \text{ minimizes } f \text{ on } X)$$

Indeed let $y \in X$ be arbitrary. Then there $\exists$ a recovery sequence $\{y_n\}$ s.t.

$$y_n \to y \quad \text{and} \quad f(y) = \lim_{n \to +\infty} f_n(y_n)$$

Then

$$f(y) = \lim_{n \to +\infty} f_n(y_n) \overset{\{f_n(y_n)\} \text{ is convergent}}{=} \liminf_{n \to +\infty} f_n(y_n) \overset{\text{def of inf}}{\geq} \liminf_{n \to +\infty} \inf_{x \in X} f_n(x)$$

$$\overset{\text{Equicoercivity}}{=} \liminf_{n \to +\infty} \left\{ \inf_{x \in K} f_n(x) \right\} \underset{\text{PROP 11.9, point ②, as K is compact}}{\geq} \inf_{x \in K} f(x) \underset{\text{by (K)}}{=} f(\hat{x})$$

and so $(*)$ holds.

② We have:

$$\inf_{x \in X} f(x) \overset{\text{By PROPOSITION 11.9 point ①, since X is open}}{\geq} \liminf_{n \to +\infty} \left\{ \inf_{x \in X} f_n(x) \right\} \overset{\text{Equicoercivity}}{=} \liminf_{n \to +\infty} \left\{ \inf_{x \in K} f_n(x) \right\}$$

$$\underset{\text{By PROPOSITION 11.9 point ②, as K is compact}}{\geq} \inf_{x \in K} f(x) \underset{\text{by (K) and (*)}}{=} \min_{x \in X} f(x)$$

proving that

$$(**) \quad \liminf_{n\to+\infty} \left\{ \inf_{x\in X} f_n(x) \right\} = \min_{x\in X} f(x)$$

Now let $\hat{x}$ be minimizer for f on X, which exists by point 1.
Let $\{x_n\}$ be a recovery sequence, i.e.

$$x_n \to \hat{x} \qquad \text{and} \qquad f(\hat{x}) = \lim_{n\to+\infty} f_n(x_n)$$

Clearly

$$\inf_{x\in X} f_n(x) \leq f_n(x_n), \qquad \forall\, n \in \mathbb{N}$$

Taking the limsup in the above yields

$$(***) \quad \limsup_{n\to+\infty} \left\{ \inf_{x\in X} f_n(x) \right\} \leq \limsup_{n\to+\infty} f_n(x_n) \underset{\{x_n\} \text{ is recovery sequence}}{=} f(\hat{x}) \underset{\hat{x} \text{ is minimizer}}{=} \min_{x\in X} f(x)$$

Therefore

$$\min_{x\in X} f(x) \overset{(**)}{=} \liminf_{n\to+\infty} \left\{ \inf_{x\in X} f_n(x) \right\} \overset{\text{property of liminf / limsup}}{\leq} \limsup_{n\to+\infty} \left\{ \inf_{x\in X} f_n(x) \right\} \overset{(***)}{\leq} \min_{x\in X} f(x)$$

concluding.

③ Let $\{x_n\}$ be a sequence of quasi-minimizers s.t. $x_{n_k} \to \hat{x}$. Then

$$f(\hat{x}) \overset{\Gamma\text{-liminf inequality}}{\leq} \liminf_{k\to+\infty} f_{n_k}(x_{n_k}) = \liminf_{k\to+\infty} \left\{ \underbrace{f_{n_k}(x_{n_k}) - \inf_{x\in X} f_{n_k}(x)}_{\to 0 \text{ by assumption}} + \inf_{x\in X} f_{n_k}(x) \right\}$$

$$= \liminf_{k\to+\infty} \left\{ \inf_{x\in X} f_{n_k}(x) \right\} \overset{\text{point ② of this Theorem}}{=} \min_{x\in X} f(x)$$

showing that $\hat{x}$ minimizes f over X. □

EXAMPLE 11.13 Consider the functionals $F_n : C^1[0,1] \to \mathbb{R}$ defined by

$$F_n(u) := \int_0^1 n\dot{u}^2 + (u - \arctan x)^2 \, dx$$

QUESTION What is the limit of $M_n := \inf\{F_n(u) : u \in C^1[0,1]\}$.

Extend F_n to $L^2(0,1)$ by setting $F_n := +\infty$ in $L^2 \setminus C^1$. Thus

$$M_n = \inf\{F_n(u) \mid u \in L^2(0,1)\}.$$

CLAIM $F_n \xrightarrow{\Gamma} F$ in $L^2(0,1)$, with

$$F(u) := \begin{cases} \int_0^1 (u - \arctan x)^2 \, dx, & \text{if } u \text{ is constant} \\ +\infty, & \text{otherwise in } L^2(0,1) \end{cases}$$

<u>Proof of CLAIM</u> Note that $F_n = G_n + H$ with

$$G_n(u) := \begin{cases} \int_0^1 n \dot{u}^2 \, dx & \text{if } u \in C^1[0,1] \\ +\infty & \text{otherwise} \end{cases}, \qquad H(u) := \int_0^1 (u - \arctan x)^2 \, dx$$

Clearly H is continuous in $L^2(0,1)$. Therefore, by PROPOSITION 11.4, it is sufficient to compute the Γ-limit of G_n. We have that

$$G_n \xrightarrow{\Gamma} G, \qquad \text{with} \qquad G(u) := \begin{cases} 0, & \text{if } u \text{ is constant} \\ +\infty, & \text{otherwise in } L^2(0,1) \end{cases}$$

- Γ-liminf inequality: suppose $u_n \to u$ in $L^2(0,1)$. We need to show

$$(*) \qquad G(u) \leq \liminf_{n \to +\infty} G_n(u_n).$$

WLOG we can assume the RHS to be finite, so there $\exists$ a subsequence s.t.

$$G_{n_k}(u_{n_k}) \leq M, \qquad \forall \, k \in \mathbb{N}.$$

This means

$$\int_0^1 \dot{u}_{n_k}^2 \, dx \leq \frac{M}{n_k},$$

which implies

$$\dot{u}_{n_k} \to 0 \quad \text{strongly in } L^2(0,1).$$

Since we are assuming $u_n \to u$ strongly in $L^2(0,1)$, REMARK 7.17 implies

$$u_{n_k} \to u \quad \text{strongly in } W^{1,2}(0,1), \qquad \text{with } \dot{u} = 0 \quad \text{weakly}.$$

Thus $u \in W^{1,2}(0,1)$ with $\dot{u} \in C[0,1]$. Therefore $u \in C^1[0,1]$ by PROPOSITION 7.22. Hence the relationship $\dot{u} = 0$ also holds in the classical sense (as the weak derivative of a differentiable function coincides with the classical one). Since $[0,1]$ is connected then

$$u \in C^1[0,1], \quad \dot{u} = 0 \quad \Longrightarrow \quad u = \text{constant}$$

Thus $G(u) = 0$ by definition and $(*)$ holds (being $G_n \geq 0$).

- Γ-limsup inequality: Let $u \in L^2(0,1)$. We need to construct a recovery sequence.

  - If u is not constant, then $G(u) = +\infty$. Thus setting $u_n := u$, $\forall n \in \mathbb{N}$ we get $u_n \to u$ and, trivially,

$$\limsup_{n \to +\infty} G_n(u_n) \leq +\infty = G(u).$$

  - If u is constant, then $G(u) = 0$. Again set $u_n := u$, $\forall n \in \mathbb{N}$. Then $u_n \to u$. Moreover, as u is constant, then $u \in C^\infty[0,1]$ and $\dot{u} = 0$. Therefore

$$G_n(u_n) = G_n(u) = \int_0^1 n \dot{u}^2 \, dx = 0, \qquad \forall n \in \mathbb{N}$$

and the Γ-limsup inequality trivially holds.

Then $G_n \xrightarrow{\Gamma} G$ and so $F_n = G_n + H \xrightarrow{\Gamma} G + H = F$, by PROPOSITION 11.4. $\square$

In order to apply THEOREM 11.12, we also need to show that the sequence of functionals F_n is EQUICOERCIVE in $L^2(0,1)$.

CLAIM $\{F_n\}$ is EQUICOERCIVE in $L^2(0,1)$.

Proof of Claim By REMARK 11.11 it is sufficient to show $\exists$ of $M \in \mathbb{R}$ s.t.

$$K := \{ u \in L^2(0,1) \mid F_n(u) \leq M \}$$

is non-empty and pre-compact. First of all, note that

$$F_n(0) = \int_0^1 (\arctan x)^2 \, dx \leq \left(\frac{\pi}{2}\right)^2 < 10 \quad , \quad \forall n \in \mathbb{N}.$$

We then choose $M := 10$, so that $K \neq \emptyset$. We are left to show that K is pre-compact in $L^2(0,1)$. Indeed,

$$F_n(u) \leq 10 \overset{\text{def of } F_n}{\Longrightarrow} \begin{cases} \displaystyle\int_0^1 \dot{u}^2 \, dx \leq \frac{10}{n} \\ \\ \displaystyle\int_0^1 (u - \arctan x)^2 \, dx \leq 10 \end{cases} \Rightarrow \|u\|_{W^{1,2}} \leq C$$

for some $C > 0$ not depending on n and on u. Thus

$$K = \{ u \in L^2(0,1) \mid F_n(u) \leq 10 \} \subseteq \tilde{K} := \{ u \in W^{1,2}(0,1) \mid \|u\|_{W^{1,2}} \leq C \}$$

Note that $\tilde{K}$ is compact in $L^2(0,1)$, thanks to the compact embedding $W^{1,2}(0,1) \hookrightarrow L^2(0,1)$ of THEOREM 7.27.

Therefore K is pre-compact, since $\overline{K}$ is closed and contained in the compact $\tilde{K}$. $\square$

Thus we have shown

(i) $\{F_n\}$ is EQUICOERCIVE in $L^2(0,1)$

(ii) $F_n \xrightarrow{\Gamma} F$ in $L^2(0,1)$

From THEOREM 11.12 we then get

$$\inf_{u \in L^2(0,1)} F_n(u) \longrightarrow \min_{u \in L^2(0,1)} F(u),$$

that is,

$$M_n \to M := \min_{u \in L^2(0,1)} F(u)$$

Since $F(u) < +\infty$ if and only if u is constant, then

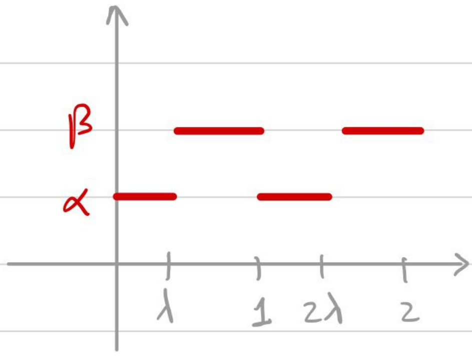
$$M = \min \left\{ \int_0^1 (\lambda - \arctan x)^2 \, dx \;\middle|\; \lambda \in \mathbb{R} \right\}$$

which can be computed explicitly.

APPLICATION: HOMOGENIZATION PROBLEMS

DEFINITION 11.14 For $\alpha, \beta \in \mathbb{R}$, $\lambda \in (0,1)$ define

$$A(x) := \begin{cases} \alpha & \text{if } x \in [0, \lambda) \\ \beta & \text{if } x \in [\lambda, 1) \end{cases}$$



Extend A to $\mathbb{R}$ by periodicity. For $n \in \mathbb{N}$ set

$$A_n(x) := A(nx)$$

FACT $A_n(x) \rightharpoonup \underbrace{\lambda \alpha + (1-\lambda)\beta}_{\text{Average of } A \text{ in } [0,1]}$ weakly in $L^p(a,b)$, $\forall\, 1 \leq p < +\infty$

Define $F_n : C^1[a,b] \to \mathbb{R}$ by

$$F_n(u) := \int_a^b A_n(x)\, \dot{u}(x)\, dx$$

Extend F_n to $+\infty$ on $L^2(a,b) \setminus C^1[a,b]$. Define $F : L^2(a,b) \to \overline{\mathbb{R}}$ by

$$F(u) := \begin{cases} \int_a^b \dot{u}^2\, dx & \text{if } u \in W^{1,2}(a,b) \\ +\infty & \text{otherwise} \end{cases}$$

One might expect $F_n \xrightarrow{\Gamma} cF$ with $c = \lambda\alpha + (1-\lambda)\beta$. However this is <u>FALSE</u>

THEOREM 11.15 Suppose $\alpha, \beta > 0$. Consider F_n, F as above. Then

$$F_n \xrightarrow{\Gamma} cF, \qquad c := \frac{1}{\frac{\lambda}{\alpha} + \frac{1-\lambda}{\beta}}$$

(Harmonic mean of A in $[0,1]$: $\int_0^1 \frac{1}{A(x)}\, dx$)

In order to prove the above, consider the following:

CELL - PROBLEM For $\ell > 0$, consider the problem:

$$\min \left\{ \int_0^1 A(x) \dot{u}^2 \, dx \;\middle|\; u(0) = 0,\ u(1) = \ell \right\}$$

This is called cell-problem because A has only one oscillation in $[0,1]$

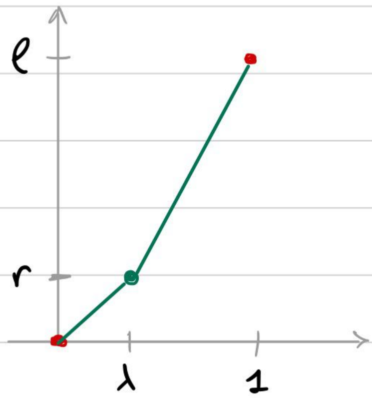
LEMMA 11.16 Let $\alpha, \beta > 0$. The cell-problem has solution

$$\min \left\{ \int_0^1 A(x) \dot{u}^2 \, dx \;\middle|\; u(0) = 0,\ u(1) = \ell \right\} = c\ell^2, \qquad c := \frac{1}{\frac{\lambda}{\alpha} + \frac{1-\lambda}{\beta}}$$

Proof Since $A = \alpha$ in $[0, \lambda)$ and $A = \beta$ in $[\lambda, 1)$, the separate problems in $[0, \lambda]$ and $[\lambda, 1]$ become

$$\min \left\{ \alpha \int_0^\lambda \dot{u}^2 \, dx \;\middle|\; u(0) = 0,\ u(\lambda) = r \right\}$$

$$\min \left\{ \beta \int_\lambda^1 \dot{u}^2 \, dx \;\middle|\; u(\lambda) = r,\ u(1) = \ell \right\}$$



We already know that the above problems are solved by straight lines u_1, u_2 respectively. In particular

$$\dot{u}_1 = \frac{r}{\lambda}, \qquad \dot{u}_2 = \frac{\ell - r}{1 - \lambda}$$

Define

$$u_r(x) := \begin{cases} u_1 & \text{if } x \in [0, \lambda) \\ u_2 & \text{if } x \in [\lambda, 1) \end{cases}$$

It is easy to show that

$$(*) \quad \min\left\{ \int_0^1 A(x)\,\dot{u}^2\,dx \;\middle|\; u(0)=0,\ u(1)=\ell \right\} = \min\left\{ \int_0^1 A(x)\,\dot{u}_r^2\,dx \;:\; r\in\mathbb{R} \right\}$$

Now

$$\int_0^1 A(x)\,\dot{u}_r^2\,dx = \int_0^\lambda A(x)\,\dot{u}_1^2\,dx + \int_\lambda^1 A(x)\,\dot{u}_2^2\,dx = \alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda}$$

so that

$$\min_{r\in\mathbb{R}} \int_0^1 A(x)\,\dot{u}_r^2\,dx = \min_{r\in\mathbb{R}} \left\{ \alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda} \right\}$$

Now

$$\alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda} = Ar^2 + Br + C, \qquad \begin{cases} A = \dfrac{\alpha}{\lambda} + \dfrac{\beta}{1-\lambda} \\[2ex] B = -\dfrac{2\ell\beta}{1-\lambda} \\[2ex] C = \dfrac{\beta}{1-\lambda}\,\ell^2 \end{cases}$$

which is minimized at $r = -\dfrac{B}{2A}$. Substituting into Ar^2+Br+C, we obtain the minimum $-\dfrac{B^2}{4A}+C$, i.e.,

$$\min_{r\in\mathbb{R}} \left\{ \alpha\frac{r^2}{\lambda} + \beta\frac{(\ell-r)^2}{1-\lambda} \right\} = -\frac{B^2}{4A} + C = \frac{\alpha\beta}{\lambda\beta+(1-\lambda)\alpha}\,\ell^2 = c\,\ell^2$$

Recalling $(*)$, we conclude. $\square$

REMARK Solving the cell-problem is equivalent to solving

$$\min\left\{ c\int_0^1 \dot{u}^2\,dx \;\middle|\; u(0)=0,\ u(1)=e \right\}, \qquad c := \frac{1}{\frac{\lambda}{\alpha}+\frac{1-\lambda}{\beta}}$$

(Indeed, the solution to the above is given by the straight line $u(x)=ex$, so that $c\int_0^1 \dot{u}^2 dx = ce^2$)

LEMMA 11.17 (RESCALED CELL-PROBLEM)

The rescaled cell-problem satisfies

Harmonic average of A

$$\min\left\{ \int_{k/n}^{\frac{k+1}{n}} A(nx)\,\dot{u}^2\,dx \;\middle|\; u\left(\tfrac{k}{n}\right)=A,\ u\left(\tfrac{k+1}{n}\right)=B \right\} = cn\,(B-A)^2$$

$A(nx)$ values α in $\left[\frac{k}{n}, \frac{k}{n}+\frac{\lambda}{n}\right)$ and β in $\left[\frac{k}{n}+\frac{\lambda}{n}, \frac{k+1}{n}\right)$

Thus $A(nx)$ has only one oscillation in $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, and this is still a cell-problem

Proof Same as LEMMA 11.16. $\square$

REMARK Solving the rescaled cell-problem is equivalent to solving

$$\min\left\{ c\int_{k/n}^{\frac{k+1}{n}} \dot{u}^2\,dx \;\middle|\; u\left(\tfrac{k}{n}\right)=A,\ u\left(\tfrac{k+1}{n}\right)=B \right\}, \qquad c := \frac{1}{\frac{\lambda}{\alpha}+\frac{1-\lambda}{\beta}}$$

(Indeed, the solution to the above is given by the straight line u with

$$\dot{u} = \frac{B-A}{\frac{k+1}{n}-\frac{k}{n}} = n(B-A)$$

so that

$$c\int_{k/n}^{\frac{k+1}{n}} \dot{u}^2\,dx = c\,n^2(B-A)^2\cdot\frac{1}{n} = cn\,(B-A)^2$$

)

Γ-LIMINF INEQUALITY FOR THEOREM 11.15

We now sketch the proof of the Γ-liminf inequality in THEOREM 11.15.
Let $u_n \to u$ strongly in $L^2(a,b)$. We need to prove that

$$(*) \qquad c\,F(u) \leq \liminf_{n \to +\infty} F_n(u_n)$$

WLOG assume RHS finite, i.e., $\exists$ a subsequence s.t.

$$F_{n_k}(u_{n_k}) \leq M \quad , \qquad \forall\, k \in \mathbb{N}.$$

In particular $\{u_{n_k}\} \subseteq C^1[a,b]$ and

$$\int_a^b A_{n_k}(x)\, \dot{u}_{n_k}^2 \, dx \leq M \quad , \qquad \forall\, k \in \mathbb{N}.$$

Now $A_{n_k} \geq \min\{\alpha, \beta\} > 0$, from which we deduce that $\{\dot{u}_{n_k}\}$ is bounded in $L^2(a,b)$. Thus, $\exists\, v \in L^2(a,b)$ s.t.

$$\dot{u}_{n_{k_j}} \rightharpoonup v \quad \text{weakly in } L^2(a,b)$$

As $u_n \to u$ in $L^2(a,b)$, from REMARK 7.18 we conclude that

$$u_{n_{k_j}} \rightharpoonup u \quad \text{weakly in } W^{1,2}(a,b)$$

Since the above limit does not depend on the subsequence, we get convergence along the whole sequence, i.e.,

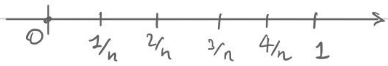
$$(W) \qquad u_n \rightharpoonup u \quad \text{weakly in } W^{1,2}(a,b)$$

The compact embedding $W^{1,2}(a,b) \hookrightarrow C[a,b]$ (see THEOREM 7.27) implies

$$(U) \qquad u_n \to u \quad \text{uniformly in } [a,b] \quad , \qquad \{u_n\}, u \text{ continuous}$$

• **Step 1:** Assume $u_n \to u$ uniformly in $[0,1]$.
We want to prove ⊛ localized to $[0,1]$ (i.e. $a=0$, $b=1$)

Divide $[0,1]$ in subintervals $\left[\frac{k}{n}, \frac{k+1}{n}\right]$. Then



$$\int_0^1 A(nx)\, \dot{u}_n^2 \, dx = \sum_{k=0}^{n-1} \underbrace{\int_{\frac{k}{n}}^{\frac{k+1}{n}} A(nx)\, \dot{u}_n^2 \, dx}$$

u_n is competitor for RESCALED CELL-PROBLEM
WITH $A = u_n\left(\frac{k}{n}\right)$, $B = u_n\left(\frac{k+1}{n}\right)$

$$\text{(LEMMA 11.17)} \geq \sum_{k=0}^{n-1} c\, n \left[u_n\left(\frac{k+1}{n}\right) - u_n\left(\frac{k}{n}\right) \right]^2$$

$$\left(n \sum \alpha_k^2 \geq \left(\sum \alpha_k\right)^2 \right) \geq c \left[\sum_{k=0}^{n-1} u_n\left(\frac{k+1}{n}\right) - u_n\left(\frac{k}{n}\right) \right]^2$$

$$= c \left[u_n(1) - u_n(0) \right]^2$$

As $u_n \to u$ uniformly, we get

$$\liminf_{n \to +\infty} \int_0^1 A(nx)\, \dot{u}_n^2 \, dx \geq \liminf_{n \to +\infty} c \left[u_n(1) - u_n(0) \right]^2$$

$$(u_n \to u \text{ uniformly}) = c \left[u(1) - u(0) \right]^2$$

NOTE The above would be ⊛ if $u(x) = mx + q$ line, since

$$c \int_0^1 \dot{u}^2 \, dx = c\, m^2 = c \left[u(1) - u(0) \right]^2$$

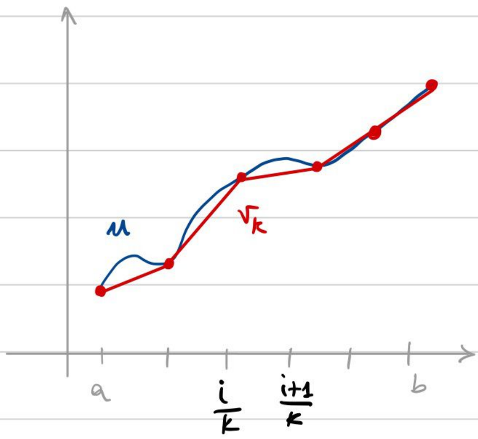
• **Step 2:** Assume $u \in L^2(\tilde a, \tilde b)$ with $\tilde a < \tilde b$ arbitrary.
Suppose $u_n \to u$ uniformly in $[\tilde a, \tilde b]$. By a rescaling argument, and proceeding as in STEP 1, one can show

$$(L) \qquad \liminf_{n\to+\infty} \int_{\tilde a}^{\tilde b} A_n(x)\,\dot u_n^2\,dx \geq c\,(\tilde b - \tilde a)\left[\frac{u(\tilde b) - u(\tilde a)}{\tilde b - \tilde a}\right]^2$$

NOTE Again, (L) is ✱ if $u(x) = mx + q$

• **Step 3:** General case. Assume $u_n \to u$ in $L^2(a,b)$. Then WLOG (W)-(U) hold and u_n, u are continuous.

Divide (a,b) in k equal parts, $k \in \mathbb{N}$ (suppose $\frac{b-a}{k} \in \mathbb{N}$)
Let v_k be the linear interpolation of u on the grid (recall u continuous)



By applying (L) to u_n on $\left[\frac{i}{k}, \frac{i+1}{k}\right]$ we get
(this is possible as $u_n \to u$ uniformly)

$$\liminf_{n\to+\infty} \int_{\frac{i}{k}}^{\frac{i+1}{k}} A(nx)\,\dot u_n^2\,dx \overset{(L)}{\geq} c\,\frac{1}{k}\left(\frac{u\left(\frac{i+1}{k}\right) - u\left(\frac{i}{k}\right)}{\frac{1}{k}}\right)^2$$

$$= c\int_{\frac{i}{k}}^{\frac{i+1}{k}} \dot v_k^2\,dx \qquad \left(\text{as } \dot v_k = \frac{u\left(\frac{i+1}{k}\right) - u\left(\frac{i}{k}\right)}{1/k} \text{ on } \left[\frac{i}{k}, \frac{i+1}{k}\right]\right)$$

Summing over i we find

$$✱✱ \qquad \liminf_{n\to+\infty} F_n(u_n) \geq c\,F(v_k), \qquad \forall k \in \mathbb{N}$$

Now, one can check that as the grid width goes to zero, we have

$$v_k \to u \quad \text{uniformly}, \quad \text{and} \quad F(v_k) \nearrow F(u)$$

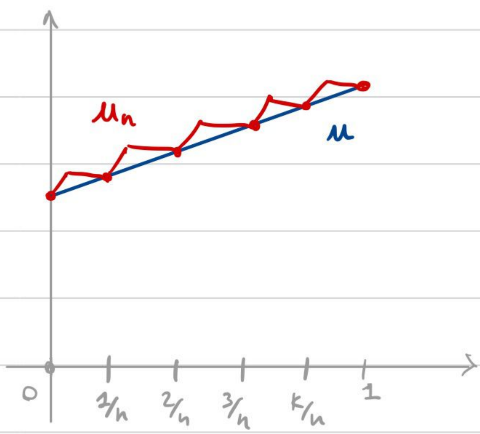
Thus, taking the sup for $k \in \mathbb{N}$ in (**) yields (*), concluding the Γ-liminf inequality.

Γ-LIMSUP INEQUALITY FOR THEOREM 11.15

We now sketch the proof of the Γ-limsup inequality in THEOREM 11.15

- **Step 1** : $u = \ell x + q$ in $[0,1]$. Then we can chose the recovery sequence u_n in the following way:

Divide $[0,1]$ in sub-intervals $\left[\frac{k}{n}, \frac{k+1}{n}\right]$. In each sub-int. u_n is the solution to the rescaled cell-problem with data $A = u\left(\frac{k}{n}\right)$, $B = u\left(\frac{k+1}{n}\right)$



In this way the energy in each $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ is

$$\int_{k/n}^{\frac{k+1}{n}} A(nx)\, \dot{u}_n^2 \, dx = c\, n \left[u\left(\frac{k+1}{n}\right) - u\left(\frac{k}{n}\right) \right]^2 = c \int_{k/n}^{\frac{k+1}{n}} \dot{u}^2 \, dx$$

u_n solution to rescaled cell-problem (LEMMA 11.17)

u straight line

and the total energy becomes:

$$F_n(u_n) = \sum_{k=0}^{n-1} \int_{k/n}^{\frac{k+1}{n}} A(nx)\, \dot{u}_n^2 \, dx$$

$$= c \sum_{k=0}^{n-1} \int_{k/n}^{\frac{k+1}{n}} \dot{u}^2 \, dx \quad = c\, F(u)$$

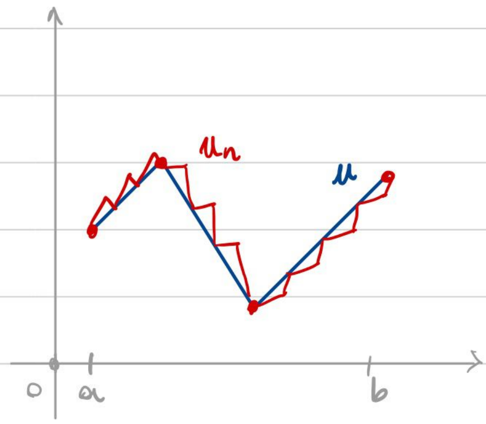
Thus $F_n(u_n) \to c\, F(u)$ trivially. One can also show that, as the grid refines,

$$u_n \to u \quad \text{uniformly in } [0,1],$$

concluding the Γ-limsup inequality.

• **Step 2:** u piecewise affine in $[a,b]$.

To obtain u_n just divide $[a,b]$ into the sub-intervals in which u is affine and in each of those define u_n as in STEP 1.



• **Step 3:** Let $u \in L^2(a,b)$ be arbitrary.

REMARK 11.18 In general, it is sufficient to show the Γ-limsup inequality for elements in D, where $D \subseteq X$ is an energy dense set WRT the Γ-limit.

(This is easily proven with a diagonal argument)

Choose D as the set of PIECEWISE AFFINE FUNCTIONS. Then D is energy dense WRT cF (easy check). The Γ-limsup inequality holds in D by STEP 2. Therefore we conclude the Γ-limsup inequality in $L^2(a,b)$ by REMARK 11.18.

EXAM INFO

- ORAL EXAM ON TOPICS SEEN DURING THE COURSE
  (I WILL REFER TO THE SUMMARY ON THE COURSE WEB PAGE)

- PLEASE, SEND ME AN EMAIL WITH SUGGESTED DATES
  (silvio.fanzon@uni-graz.at)

- EXAM HELD ONLINE
  (IF OPTION IS AVAILABLE)