EXTENSION BY RELAXATION : CONVEX CASE

LESSON 14
16 JUNE 2021

Setting : $(\hat{X}, d)$ metric space, $X \subseteq \hat{X}$ and $f : X \to \overline{\mathbb{R}}$.

QUESTION Find $\hat{f} : \hat{X} \to \overline{\mathbb{R}}$ which extends f in a meaningful way.

IDEA Extend f on $\hat{X}$ by setting

$$(*) \qquad \hat{f}(x) := \begin{cases} f(x) & \text{if } x \in X \\ +\infty & \text{if } x \in \hat{X} \setminus X \end{cases}$$

Then consider $\hat{f} := \overline{f}$. In the following f is always extended according to $(*)$

EXAMPLE $\hat{X} = L^2(a,b)$, $X = C^1[a,b]$, $F : X \to \overline{\mathbb{R}}$ by

$$F(u) := \int_a^b \dot{u}^2 \, dx \, , \qquad \forall \, u \in X .$$

Extend F to $+\infty$ on $\hat{X} \setminus X$. Then set $\hat{F} := \overline{F}$. (relax in L^2)

CLAIM We have that

weak derivative

$$\hat{F}(u) = G(u) := \begin{cases} \int_a^b \dot{u}^2 \, dx & \text{if } u \in H^1(a,b) \\ +\infty & \text{if } u \in L^2 \setminus H^1 \end{cases}$$

(proof left as exercise. One can employ STRATEGY 2 in this case)

IN GENERAL We want to compute relaxation for $F : C^1[a,b] \to \overline{\mathbb{R}}$

$$F(u) := \int_a^b \psi(\dot{u})\, dx , \qquad \psi : \mathbb{R} \to \mathbb{R} , \quad \psi = \psi(\xi)$$

Under some assumptions the relaxation of F in $L^p(a,b)$ is given by

$$\hat{F} : L^p(a,b) \to \overline{\mathbb{R}} , \qquad \hat{F}(u) := \begin{cases} \displaystyle\int_a^b \psi(\dot{u})\, dx , & \text{if } u \in W^{1,p}(a,b) \\ +\infty & \text{otherwise in } L^p(a,b) \end{cases}$$

THEOREM 10.19 Consider F, $\hat{F}$ as above. Assume $\psi : \mathbb{R} \to \mathbb{R}$ is

1. Convex
2. $\exists\, A > 0$, $B \in \mathbb{R}$, $p \in (1, +\infty)$ s.t.

$$\psi(\xi) \geq A |\xi|^p - B , \qquad \forall \xi \in \mathbb{R} .$$

Then $\overline{F} = \hat{F}$ in $L^p(a,b)$.

Proof We use STRATEGY 2 (PROP 10.18). We need to show that:

1. $\hat{F}$ is LSC in $L^p(a,b)$
2. $\hat{F}(u) \leq F(u)$, $\forall u \in L^p(a,b)$ (Here $F(u) := +\infty$ if $u \notin C^1[a,b]$)
3. $\exists\, D \subseteq L^p(a,b)$ Energy Dense WRT $\hat{F}$, s.t.

$\forall u \in D$, $\exists \{u_n\} \subseteq C^1[a,b]$ s.t. $u_n \to u$ and $\displaystyle\limsup_{n \to +\infty} F(u_n) \leq \hat{F}(u)$

Strongly in $L^p(a,b)$

Checking 1: Need to show that if $u_n \to u$ in $L^p(a,b)$ then

$$\circledast \qquad \hat{F}(u) \le \liminf_{n\to+\infty} \hat{F}(u_n).$$

If RHS is $+\infty$ then $\circledast$ is trivial. Then WLOG we can assume that

$$\hat{F}(u_n) \le M, \quad \forall n \in \mathbb{N}.$$

From the growth assumption on ψ we get

$$\int_a^b A|\dot{u}_n|^p - B \, dx \le \hat{F}(u_n) \le M$$

so that

$$\int_a^b |\dot{u}_n|^p \, dx \le \frac{M + (b-a)B}{A}$$

proving that $\{\dot{u}_n\}$ is bounded in $L^p(a,b)$. Thus, up to subsequences

$$\dot{u}_n \rightharpoonup v \quad \text{weakly in } L^p(a,b)$$

As $u_n \to u$ strongly in $L^p(a,b)$, in particular we get

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(a,b)$$

Thus, from REMARK 7.18 (trivially adaptable to $W^{1,p}$ case) we get

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(a,b)$$

Now $\circledast$ can be shown as in THEOREM 9.9 (if we assume ψ is C^1 and growth of ψ from above). In general, see THM 3.6 in BUTTAZZO, GIAQUINTA, HILDEBRANDT

Checking 2 : This is obvious by definition of F, $\hat{F}$, and by the fact that weak derivatives coincide with classical ones for maps in $C^1[a,b]$.

Checking 3 : Set

$$D := \{ u : [a,b] \to \mathbb{R} \mid u \text{ continuous and piecewise linear} \}$$

CLAIM D is Energy Dense WRT $\hat{F}$

[Given $\hat{u} \in L^p(a,b)$ we need to find $\{u_n\} \subseteq D$ s.t.

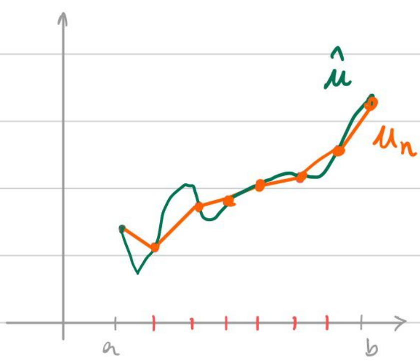
$$(*) \qquad u_n \to \hat{u} \text{ in } L^p \quad \text{and} \quad \hat{F}(u_n) \to \hat{F}(\hat{u})$$

- If $\hat{u} \notin W^{1,p}(a,b)$ then $\hat{F}(\hat{u}) = +\infty$. Now it is easy to approximate u in L^p with a sequence in D and obtain $(*)$ by LSC of $\hat{F}$:

$$+\infty = \hat{F}(\hat{u}) \leq \liminf_{n \to +\infty} \hat{F}(u_n) \implies \lim_{n \to +\infty} \hat{F}(u_n) = +\infty.$$

- If $\hat{u} \in W^{1,p}(a,b)$ and $\hat{F}(\hat{u}) = +\infty$, then proceed as above.

- If $\hat{u} \in W^{1,p}(a,b)$ and $\hat{F}(\hat{u}) < +\infty$: by THEOREM 7.19 we know that $\hat{u} \in C[a,b]$. Then construct u_n as in picture



Divide $[a,b]$ in sub-intervals I_i of amplitude $1/n$.
Define u_n by linear interpolation of values of $\hat{u}$ on the grid.

As the mesh-size goes to zero as $n \to +\infty$ and $\hat{u}$ is uniformly continuous in $[a,b]$ we get

$$u_n \to \hat{u} \quad \text{uniformly in } [a,b] \qquad \text{(easy check)}$$

Then in particular

$$u_n \to \hat{u} \quad \text{strongly in } L^p(a,b)$$

Moreover, it holds that

$$(**) \qquad \hat{F}(u_n) \leq \hat{F}(\hat{u}) \quad , \quad \forall n \in \mathbb{N} .$$

Indeed

$$\hat{F}(u_n) \overset{\text{def of } u_n}{=} \sum_{k=1}^{N} \int_{I_k} \Psi(\dot{u}_n)\, dx$$

Now, consider the problem:

$$\text{(P)} \qquad \min \left\{ \int_{I_k} \Psi(\dot{u})\, dx \;\middle|\; u \in W^{1,p}(I_k) \;,\; u_{|\partial I_k} = \hat{u}_{|\partial I_k} \right\}$$

Since $\Psi = \Psi(\xi)$, and Ψ is convex, one immediately sees that the straight line solves (P) (by Jensen's Inequality THEOREM 6.8). Thus

$$(***) \quad \int_{I_k} \Psi(\dot{u}_n)\, dx \leq \int_{I_k} \Psi(\dot{u})\, dx \quad , \quad \forall\, u \in W^{1,p}(I_k) \text{ s.t. } u_{|\partial I_k} = \hat{u}_{|\partial I_k}$$

Since $\hat{u}$ satisfies the Dirichlet BC, we get

$$\hat{F}(u_n) \overset{\text{def of } u_n}{=} \sum_{k=1}^{N} \int_{I_k} \Psi(\dot{u}_n)\,dx \overset{(**)}{\leq} \sum_{k=1}^{N} \int_{I_k} \Psi(\hat{u}')\,dx$$

$$= \int_a^b \Psi(\hat{u}')\,dx = \hat{F}(\hat{u})$$

so that (**) holds. Taking the limsup in (**) yields

$$\text{(LS)} \qquad \limsup_{n \to +\infty} \hat{F}(u_n) \leq \hat{F}(\hat{u}).$$

By (1) we know that $\hat{F}$ is LSC in $L^p(a,b)$, so that

$$\text{(LI)} \qquad \hat{F}(\hat{u}) \leq \liminf_{n \to +\infty} \hat{F}(u_n)$$

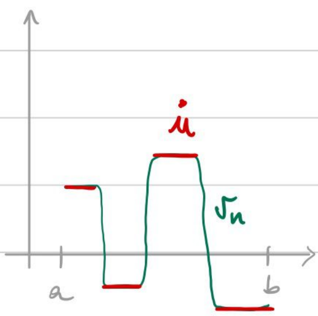
since $u_n \to u$ in $L^p(a,b)$ by construction

From (LS)–(LI) we conclude $\hat{F}(u_n) \to \hat{F}(\hat{u})$, and (*) follows.]

CLAIM $\forall u \in D$, $\exists \{u_n\} \subseteq C^1[a,b]$ s.t.

$u_n \to u$ in $L^p(a,b)$ and $\limsup_{n \to +\infty} F(u_n) \leq \hat{F}(u)$

[Usual approximation argument: for $u \in D$, we approximate $\dot{u}$ with some smooth v_n and then define u_n as the primitive of v_n.]

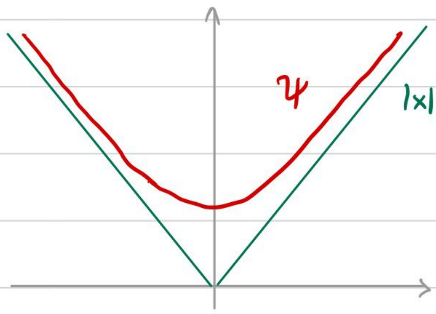


Therefore (1), (2), (3) hold and so PROP 10.18 implies $\overline{F} = \hat{F}$ in $L^p(a,b)$.

□

WARNING The thesis of THEOREM 10.19 is FALSE for $p = 1$.

[For example consider



$$\psi(\xi) := \sqrt{1+\xi^2}$$

which is CONVEX and such that

$$\psi(\xi) \geq |\xi| \quad , \qquad \forall \xi \in \mathbb{R}.$$

Consider the functional $F : C^1[-1,1] \to \mathbb{R}$

$$F(u) := \int_{-1}^{1} \sqrt{1+\dot{u}^2}\, dx$$

and $\hat{F} : W^{1,1}(-1,1) \to \mathbb{R}$

$$\hat{F}(u) := \begin{cases} \displaystyle\int_{-1}^{1} \sqrt{1+\dot{u}^2}\, dx & \text{if } u \in W^{1,1}(-1,1) \\ +\infty & \text{otherwise} \end{cases}$$

Then

$$\boxed{\overline{F} \neq \hat{F}}$$

In fact, let

$$\hat{u}(x) := \begin{cases} 0 & \text{if} \quad x \in [-1,0) \\ J & \text{if} \quad x \in [0,1] \end{cases} \quad , \qquad J \in \mathbb{R}$$

Then $\hat{F}(\hat{u}) = +\infty$, since $\hat{u} \notin W^{1,1}(-1,1)$. But $\overline{F}(\hat{u}) = |J|$.
In this case $\overline{F}$ is finite on the space $BV(-1,1)$.]

EXAMPLE Consider the functional of EXAMPLE 9.8 :

$$F : C^1[a,b] \to \mathbb{R} \quad , \qquad F(u) := \int_a^b \dot{u}^2 + \sin(u^5)\, dx$$

We can write $F = G + H$ with

$$G(u) := \int_a^b \psi(\dot{u})\, dx \quad , \qquad \psi(\xi) := \xi^2 \quad , \qquad H(u) := \int_a^b \sin(u^5)\, dx$$

ψ is convex and satisfies the growth assumption of THEOREM 10.19 with $p=2$, $A=1$, $B=0$. Therefore the relaxation of G in $L^2(a,b)$ is

$$\bar{G}(u) = \begin{cases} \int_a^b \dot{u}^2\, dx & \text{if } u \in W^{1,2}(a,b) \\ +\infty & \text{otherwise} \end{cases}$$

Also notice that H is continuous in $L^2(a,b)$. Thus $\bar{H} = H$ (exercise)
Then one can prove (exercise)

$$\bar{F} = \bar{G} + H$$

showing that

$$\bar{F}(u) = \begin{cases} \int_a^b \dot{u}^2 + \sin(u^5)\, dx & \text{if } u \in W^{1,2}(a,b) \\ +\infty & \text{otherwise in } L^2(a,b) \end{cases}$$

is the relaxation of F in $L^2(a,b)$, as anticipated in EXAMPLE 9.8.

EXTENSION BY RELAXATION : NON - CONVEX CASE

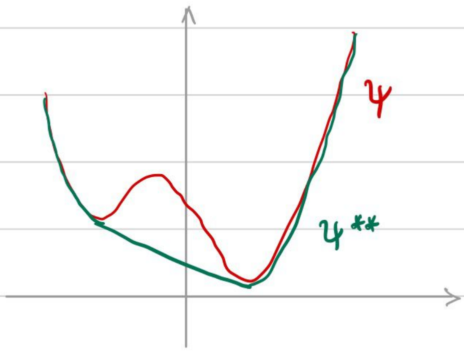
We want to generalize THEOREM 10.19 to NON-CONVEX Lagrangians.

DEFINITION 10.20 (CONVEX ENVELOPE)

Let $\psi : \mathbb{R} \to \mathbb{R}$. The CONVEX ENVELOPE of ψ is the map $\psi^{**} : \mathbb{R} \to \overline{\mathbb{R}}$

$$\boxed{\psi^{**}(x) := \sup \left\{ g(x) \mid g : \mathbb{R} \to \overline{\mathbb{R}} \text{ is convex}, \ g \leq \psi \text{ in } \mathbb{R} \right\}}$$

PROPOSITION 10.21 (PROPERTIES OF ψ^{**})



Let $\psi : \mathbb{R} \to \mathbb{R}$. Then

1. ψ^{**} is convex
2. $\psi^{**} \leq \psi$ on $\mathbb{R}$
3. The supremum in the definition of ψ^{**} is a maximum
4. We have

$$\psi^{**}(x) = \sup \left\{ mx + q \mid my + q \leq \psi(y), \ \forall y \in \mathbb{R} \right\}$$

i.e. ψ^{**} is the supremum of all lines below the graph of ψ.

(Proof is omitted)

THEOREM 10.22 Let $\psi : \mathbb{R} \to \mathbb{R}$ and define $F : C^1[a,b] \to \overline{\mathbb{R}}$ by

$$F(u) := \int_a^b \psi(\dot{u})\, dx$$

Suppose $\exists\, p \in (1, +\infty)$ and $A > 0$, $B \in \mathbb{R}$ s.t.

$$(*) \qquad \psi(\xi) \geq A|\xi|^p - B, \qquad \forall\, \xi \in \mathbb{R}.$$

Then $\overline{F} = \hat{F}$ in $L^p(a,b)$, where

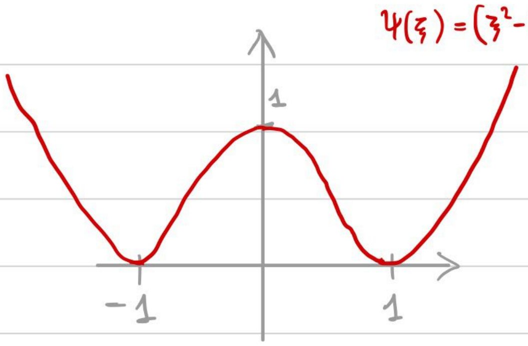
$$\hat{F}(u) := \begin{cases} \int_a^b \psi^{**}(\dot{u})\, dx , & \text{if } u \in W^{1,p}(a,b) \\ +\infty , & \text{otherwise in } L^p(a,b) \end{cases}$$

(Proof is omitted. It is similar to the proof of THEOREM 10.19)

EXAMPLE Consider the functional of EXAMPLE 5.6 : $F : C^1[a,b] \to \mathbb{R}$,

$$F(u) := \int_a^b (\dot{u}^2 - 1)^2\, dx$$

$\psi(\xi) = (\xi^2 - 1)^2$

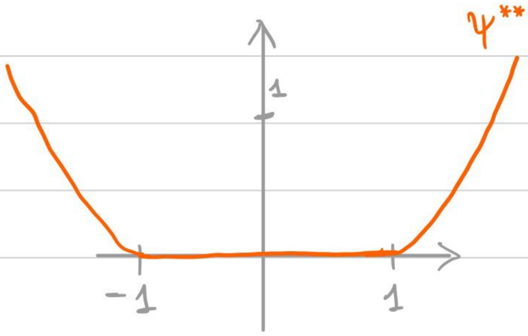


DOUBLE WELL

The Lagrangian is $\psi(\xi) := (\xi^2 - 1)^2$ which satisfies $(*)$ with $p = 4$ (for some $A > 0$, $B \in \mathbb{R}$). Thus THM 10.22 implies that the relaxation of F in $L^4(a,b)$ is given by

$$\overline{F}(u) = \begin{cases} \int_a^b \psi^{**}(\dot{u})\, dx & \text{if } u \in W^{1,4}(a,b) \\ +\infty & \text{otherwise in } L^4(a,b) \end{cases}$$

ψ^{**}



$$\psi^{**}(\xi) = \begin{cases} \psi(\xi) & \text{if } |\xi| > 1 \\ 0 & \text{if } |\xi| \leq 1 \end{cases}$$

11. GAMMA - CONVERGENCE

DEFINITION 11.1 (Γ - CONVERGENCE)

(X,d) metric space, $f_n, f : X \to \overline{\mathbb{R}}$. We say that $f_n \xrightarrow{\Gamma} f$, Γ-converges, if

① (Γ-liminf inequality) $\forall x \in X$, $\forall x_n \to x$ it holds

$$f(x) \leq \liminf_{n \to +\infty} f_n(x_n)$$

② (Γ-limsup inequality) $\forall x \in X$, $\exists x_n \to x$ such that

$$\limsup_{n \to +\infty} f_n(x_n) \leq f(x).$$

REMARK If ① and ② hold, then the limsup in ② is actually a limit.

NOTATION The sequences $\{x_n\}$ satisfying ② are called RECOVERY SEQUENCES

RELATIONSHIP WITH POINTWISE CONVERGENCE

PROPOSITION 11.2 (X,d) metric space, $f : X \to \overline{\mathbb{R}}$. Define $f_n := f$, $\forall n \in \mathbb{N}$. Then

$$f_n \xrightarrow{\Gamma} \overline{f}$$

Proof ① Γ-liminf inequality: Let $x_n \to x$. Then

$$\bar{f}(x) \overset{\text{def of } \bar{f}}{=} \inf \left\{ \liminf_{n\to+\infty} f(z_n) \ \middle|\ z_n \to x \right\}$$

$$\le \liminf_{n\to+\infty} f(x_n) \underset{f_n \equiv f}{=} \liminf_{n\to+\infty} f_n(x_n)$$

② Γ-limsup inequality: Let $x \in X$. By LEMMA 10.8 $\exists\, x_n \to x$ s.t.

$$\bar{f}(x) = \lim_{n\to+\infty} f(x_n)$$

As $f_n \equiv f$, we conclude. $\square$

REMARK PROPOSITION 11.2 implies that Γ-convergence is not related to pointwise convergence. Indeed if $f_n \equiv f$, $f \neq \bar{f}$ then

- $f_n \to f$ pointwise but $f_n \overset{\Gamma}{\not\to} f$ (because Γ-limit is unique)
- $f_n \overset{\Gamma}{\to} \bar{f}$ but $f_n \not\to \bar{f}$ pointwise (because pointwise limit is unique)

However, under additional assumptions, uniform conv. implies Γ-conv.

PROPOSITION 11.3 (X, d) metric space, $f_n, f : X \to \bar{\mathbb{R}}$. Suppose:

(i) $f_n \to f$ uniformly on compact sets of X

(ii) f is LSC.

Then $f_n \overset{\Gamma}{\to} f$.

(Proof will be left as an exercise)

STABILITY PROPERTIES

We now investigate stability properties of Γ-conv. WRT continuous perturbations.

PROPOSITION 11.4 (Stability)

(X, d) metric space, $f_n, f : X \to \bar{\mathbb{R}}$ s.t. $f_n \xrightarrow{\Gamma} f$. Assume $g : X \to \mathbb{R}$ is continuous. Then

$$f_n + g \xrightarrow{\Gamma} f + g$$

(Proof is consequence of PROPOSITION 11.5 below)

A simple generalization of the above is the following.

PROPOSITION 11.5 (Stability)

(X, d) metric space, $f_n, f : X \to \bar{\mathbb{R}}$ s.t. $f_n \xrightarrow{\Gamma} f$. Assume $g_n, g : X \to \mathbb{R}$ are such that:

(i) $g_n \to g$ uniformly on compact sets of X,

(ii) g is continuous.

Then

$$f_n + g_n \xrightarrow{\Gamma} f + g$$

(Proof will be left as an exercise)

Γ-liminf and Γ-limsup

As usual with limits, they don't always exist. For this reason one introduces notions of Γ-liminf and Γ-limsup.

DEFINITION 11.6 (X,d) metric space, $f_n : X \to \overline{\mathbb{R}}$. We define

$$\Gamma\text{-}\liminf_{n\to+\infty} f_n(x) := \inf \left\{ \liminf_{n\to+\infty} f_n(x_n) \;\middle|\; \{x_n\} \subseteq X, \; x_n \to x \right\}$$

$$\Gamma\text{-}\limsup_{n\to+\infty} f_n(x) := \inf \left\{ \limsup_{n\to+\infty} f_n(x_n) \;\middle|\; \{x_n\} \subseteq X, \; x_n \to x \right\}$$

PROPOSITION 11.7 (X,d) metric space, $f_n : X \to \overline{\mathbb{R}}$. Then Γ-liminf f_n and Γ-limsup f_n always exist and satisfy

$$\Gamma\text{-}\liminf_{n\to+\infty} f_n(x) \leq \Gamma\text{-}\limsup_{n\to+\infty} f_n(x), \qquad \forall x \in X.$$

Moreover $f_n \xrightarrow{\Gamma} f$ for some $f : X \to \overline{\mathbb{R}}$ if and only if

$$(*) \qquad \boxed{\Gamma\text{-}\liminf_{n\to+\infty} f_n(x) = \Gamma\text{-}\limsup_{n\to+\infty} f_n(x), \qquad \forall x \in X.}$$

Proof The first part of the statement is trivial. Suppose now that $f_n \xrightarrow{\Gamma} f$. Let $x_n \to x$. Then by the Γ-liminf inequality

$$f(x) \leq \liminf_{n\to+\infty} f_n(x_n)$$

and so, taking the inf for all $x_n \to x$ yields

$$(1) \qquad f(x) \leq \Gamma\text{-}\liminf_{n\to+\infty} f_n(x), \qquad \forall x \in X$$

On the other hand the Γ-limsup inequality says there $\exists\, x_n \to x$ s.t.

$$\limsup_{n\to+\infty} f_n(x_n) \leq f(x)$$

By def. of Γ-limsup we get

$$(2) \qquad \Gamma\text{-}\limsup_{n\to+\infty} f_n(x) \leq f(x) \quad , \qquad \forall\, x \in X$$

Therefore from (1)-(2) we infer

$$\Gamma\text{-}\limsup_{n\to+\infty} f_n(x) \leq \Gamma\text{-}\liminf_{n\to+\infty} f_n(x) \quad , \qquad \forall\, x\in X$$

As the other inequality was already proven in the first part of the statement, we conclude (*).

Conversely, assume (*) and set

$$f(x) := \Gamma\text{-}\liminf_{n\to+\infty} f_n(x) \overset{(*)}{=} \Gamma\text{-}\limsup_{n\to+\infty} f_n(x)$$

We want to prove that $f_n \xrightarrow{\Gamma} f$. So we need to check Γ-liminf and Γ-limsup inequalities:

- Γ-liminf ineq: Let $x_n \to x$. Then

$$\liminf_{n\to+\infty} f_n(x_n) \underset{\text{def of } \Gamma\text{-liminf } f_n}{\geq} \Gamma\text{-}\liminf_{n\to+\infty} f_n(x) \underset{\text{def of } f}{=} f(x)$$

• Γ-limsup ineq: Let $x \in X$. Since (*) holds, one can show that there there $\exists \{x_n\}$ s.t.

(3) $x_n \to x$ and $\liminf\limits_{n \to +\infty} f_n(x_n) = \limsup\limits_{n \to +\infty} f_n(x_n) = f(x)$

concluding. $\square$