10. RELAXATION

LESSON 13 - 9 JUNE 2021

LSC ENVELOPE

NOTATION In the following we denote the extended real numbers by

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$$

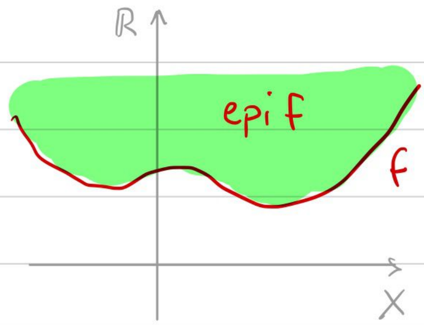
DEFINITION 10.1 (X, d) metric space, $f : X \to \overline{\mathbb{R}}$. We say that f is LOWER SEMICONTINUOUS (LSC) at $x_0 \in X$ if

$$x_n \to x_0 \text{ in } (X, d) \Rightarrow f(x_0) \leq \liminf_{n \to +\infty} f(x_n)$$

PROPOSITION 10.2 (X, d) metric space, $f : X \to \overline{\mathbb{R}}$. They are equivalent:

1. f is LSC
2. For all $x \in X$ it holds

$$f(x) \leq \liminf_{y \to x} f(y)$$



3. The epigraph of f

$$\operatorname{epi} f := \{ (x, y) \in X \times \mathbb{R} \mid y \geq f(x) \}$$

is closed in $X \times \mathbb{R}$.

4. For all $M \in \mathbb{R}$ the sublevel

$$\{ x \in X \mid f(x) \leq M \}$$

is closed in X.

(Proof is easy, but omitted)

PROPOSITION 10.3 (Sup of LSC is LSC)

(X,d) metric space, I arbitrary set of indices, $f_i : X \to \overline{\mathbb{R}}$ LSC for all $i \in I$. Then $f: X \to \overline{\mathbb{R}}$ defined by

$$f(x) := \sup \{ f_i(x) \mid i \in I \}$$

is LSC.

Proof Let $x_n \to x_0$ in X. Then

$$\liminf_{n \to +\infty} f(x_n) \geq \liminf_{n \to +\infty} f_i(x_n) \geq f_i(x_0)$$

As f is defined as the supremum

As f_i is LSC

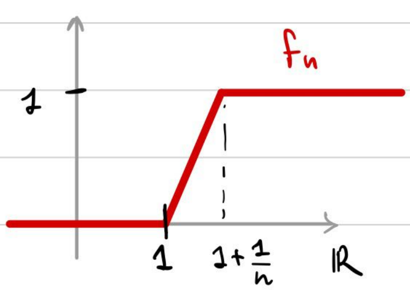
Taking the supremum for $i \in I$ allows to conclude. □

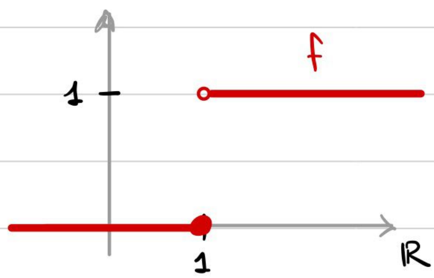
REMARK Let $f_i : X \to \mathbb{R}$ be a family of CONTINUOUS functions for $i \in I$. Then

$$(*) \quad f(x) := \sup \{ f_i(x) \mid x \in X \}$$

is in general only LSC.

For example consider f_n as in the picture. Clearly f defined by $(*)$ is not continuous.





$$f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

DEFINITION 10.4 (X,d) metric space, $f: X \to \overline{\mathbb{R}}$ a function.
The **LSC ENVELOPE** of f is the function $\hat{f}: X \to \overline{\mathbb{R}}$ defined by:

$$\hat{f}(x) := \sup \{ g(x) \mid g: X \to \overline{\mathbb{R}} \text{ is LSC}, \ g \le f \text{ on } X \}$$

REMARK

1. The LSC ENVELOPE is well-defined, since we can always consider $g \equiv -\infty$. Thus the class in which we take the sup is non-empty.

2. The LSC envelope $\hat{f}$ is LSC (by PROPOSITION 10.3)

NOTE The LSC envelope is not straightforward to compute. For this reason we introduce a more practical notion of envelope (called RELAXATION). Eventually we will prove that the two notions coincide.

RELAXATION

DEFINITION 10.5 (X,d) metric space, $f: X \to \overline{\mathbb{R}}$ a function.
The **RELAXATION** of f is the function $\overline{f}: X \to \overline{\mathbb{R}}$ defined by

$$(*) \qquad \overline{f}(x) := \inf \{ \liminf_{n \to +\infty} f(x_n) \mid \{x_n\} \subseteq X, \ x_n \to x \}$$

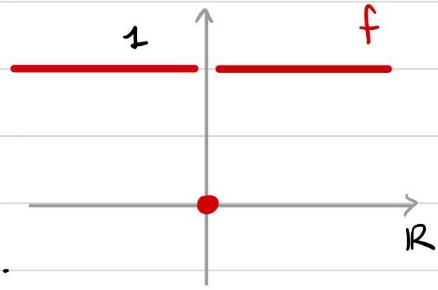
WARNING The relaxation in $(*)$ is **NOT** equivalent to

$$\overline{f}(x) \neq \liminf_{y \to x} f(y)$$

This is because the above limit does not allow to take $y \equiv x$, whereas in $(*)$ we can take $x_n \equiv x$.

For example, consider $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



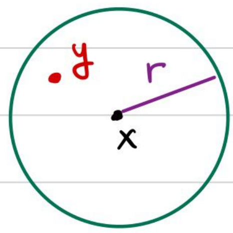
Then the relaxation is $\overline{f}(x) \equiv f(x)$.
However

$$\lim_{y \to 0} f(y) = 1$$

GOAL We aim to prove that LSC ENVELOPE and RELAXATION coincide.

LEMMA 10.6 (X, d) metric space, $f : X \to \mathbb{R}$ a function. Then

$$\boxed{\begin{gathered} \forall x \in X,\ \forall r > 0,\ \forall \varepsilon > 0,\ \exists\, y \in X \text{ s.t.} \\ d(x,y) \leq r \quad \text{and} \quad f(y) \leq \overline{f}(x) + \varepsilon \end{gathered}}$$



Proof Fix $x \in X$, $r > 0$ and $\varepsilon > 0$. By definition of Relaxation and of infimum, $\exists \{x_n\} \subseteq X$ s.t.

$$(*) \qquad x_n \to x \qquad \text{and} \qquad \liminf_{n \to +\infty} f(x_n) \leq \overline{f}(x) + \frac{\varepsilon}{2}$$

By the properties of liminf $\exists$ a subsequence $\{x_{n_k}\}$ s.t.

$$\liminf_{n \to +\infty} f(x_n) = \lim_{k \to +\infty} f(x_{n_k}).$$

From (*) we get

$$x_{n_k} \to x \qquad \text{and} \qquad \lim_{k\to+\infty} f(x_{n_k}) \leq \overline{f}(x) + \varepsilon/2 .$$

Therefore, $\exists N \in \mathbb{N}$ sufficiently large such that

$$d(x_N, x) < r, \qquad f(x_N) \leq \overline{f}(x) + \varepsilon .$$

Setting $y := x_N$ yields the thesis. □

DEFINITION 10.7 **(RECOVERY SEQUENCE)**

(X, d) metric space, $f: X \to \overline{\mathbb{R}}$. We say that $\{x_n\} \subseteq X$ is a RECOVERY SEQUENCE for f at $x \in X$ if

$$x_n \to x \qquad \text{and} \qquad \overline{f}(x) = \lim_{n\to+\infty} f(x_n)$$

LEMMA 10.8 (X, d) metric space, $f: X \to \overline{\mathbb{R}}$. For all $x \in X$ there exists a Recovery Sequence $\{x_n\} \subseteq X$.

Proof Use LEMMA 10.6 with $\varepsilon = 1/n$, $r = 1/n$ to find $y_n \in X$ s.t.

$$d(x, y_n) < \frac{1}{n}, \qquad f(y_n) \leq \overline{f}(x) + \frac{1}{n}, \qquad \forall n \in \mathbb{N}.$$

Therefore $y_n \to x$ and

$$(*) \quad \limsup_{n\to+\infty} f(y_n) \leq \limsup_{n\to+\infty} \overline{f}(x) + \frac{1}{n} = \overline{f}(x).$$

On the other hand

$$\overset{\text{def of } \bar{f}}{\bar{f}(x) =} \inf \left\{ \liminf_{n \to +\infty} f(x_n) \;\middle|\; \{x_n\} \subseteq X \;,\; x_n \to x \right\}.$$

$$\text{(Since } y_n \to x\text{)} \to \;\leq \liminf_{n \to +\infty} f(y_n) \leq \limsup_{n \to +\infty} f(y_n) \overset{(*)}{\leq} \bar{f}(x)$$

showing that $\bar{f}(x) = \lim_{n \to +\infty} f(y_n)$. Thus $\{y_n\}$ is Recovery Sequence for f at x. □

PROPOSITION 10.9 (Equivalence of LSC ENVELOPE and RELAXATION)

(X, d) metric space, $f : X \to \overline{\mathbb{R}}$ function. We have

1. $\hat{f}$ is LSC and $\hat{f}(x) \leq f(x) \quad \forall x \in X$,
2. $\bar{f}$ is LSC and $\bar{f}(x) \leq f(x) \quad \forall x \in X$,
3. $\bar{f}(x) = \hat{f}(x)$, $\quad \forall x \in X$.

Proof ① $\hat{f}$ is the supremum of LSC functions, hence it is LSC by PROP 10.3.
The inequality is obvious by definition of $\hat{f}$.

② We first show the inequality: Consider the sequence $\bar{x}_n \equiv x$. Then

$$\bar{f}(x) \overset{\text{def}}{:=} \inf \left\{ \liminf_{n \to +\infty} f(x_n) \;\middle|\; \{x_n\} \subseteq X \;,\; x_n \to x \right\}$$

$$\leq \liminf_{n \to +\infty} f(\bar{x}_n) \underset{\bar{x}_n \equiv x}{=} f(x).$$

We show that $\bar{f}$ is LSC. So let $x_n \to x_0$ be arbitrary. We want to prove

$$f(x_0) \leq \liminf_{n\to+\infty} f(x_n).$$

$\forall n \in \mathbb{N}$ apply LEMMA 10.6 with $x = x_n$, $r = 1/n$, $\varepsilon = 1/n$ to find $y_n \in X$ s.t.

$$(*) \qquad d(x_n, y_n) < \frac{1}{n}, \qquad f(y_n) \leq \bar{f}(x_n) + \frac{1}{n}, \qquad \forall n \in \mathbb{N}$$

Since $x_n \to x_0$, the first condition implies $y_n \to x_0$. Therefore

$$\bar{f}(x_0) \overset{\text{def of } \bar{f}}{=} \inf \left\{ \liminf_{n\to+\infty} f(z_n) \;\middle|\; \{z_n\} \subseteq X,\ z_n \to x_0 \right\}$$

$$(\text{As } y_n \to x_0) \to \leq \liminf_{n\to+\infty} f(y_n) \overset{(*)}{\leq} \liminf_{n\to+\infty} \left[\bar{f}(x_n) + \frac{1}{n} \right]$$

$$(\text{Property of liminf}) \leq \liminf_{n\to+\infty} \bar{f}(x_n) + \liminf_{n\to+\infty} \frac{1}{n} = \liminf_{n\to+\infty} \bar{f}(x_n),$$

showing that $\bar{f}$ is LSC.

3 • $\bar{f}(x) \geq \hat{f}(x)$: Let $x_n \to x$ be arbitrary. Then by 1

$$\liminf_{n\to+\infty} f(x_n) \underset{f \geq \hat{f}}{\geq} \liminf_{n\to+\infty} \hat{f}(x_n) \underset{\hat{f} \text{ is LSC}}{\geq} \hat{f}(x)$$

Taking the infimum for all sequences $\{x_n\} \subseteq X$ s.t. $x_n \to x$, we obtain the thesis.

• $\hat{f}(x) \geq \bar{f}(x)$: $\bar{f}$ is LSC and $\bar{f} \leq f$ by ②, thus

$$\hat{f}(x) \overset{\text{def of } \hat{f}}{=} \sup \{ g(x) \mid g: X \to \bar{\mathbb{R}},\ g \text{ LSC},\ g \leq f \text{ on } X \} \overset{\text{As } \bar{f} \text{ is competitor}}{\geq} \bar{f}(x)$$

$\square$

NOTE In the following $\bar{f}$ and $\hat{f}$ will be used interchangeably, depending on which is the most convenient.

RELATIONSHIP BETWEEN inf / min f AND inf / min $\bar{f}$

The next proposition shows why RELAXATION and LSC ENVELOPE are useful.

PROPOSITION 10.10 (X, d) metric space, $f: X \to \bar{\mathbb{R}}$ function. Then

$$\boxed{\inf_{x \in X} f(x) = \inf_{x \in X} \bar{f}(x) = \inf_{x \in X} \hat{f}(x)}$$

Proof Since $\bar{f} = \hat{f}$ by PROPOSITION 10.9, we only need to show the first equality.

$\boxed{\geq}$ This is clear, since $f \geq \bar{f}$ by PROPOSITION 10.9

$\boxed{\leq}$ Let $\{x_n\}$ be an infimizing sequence for $\bar{f}$, i.e.,

$$\overline{f}(x_n) \to \inf_{x \in X} \overline{f}(x) .$$

For all $n \in \mathbb{N}$ apply LEMMA 10.6 with $x = x_n$, $r = 1$, $\varepsilon = 1/n$, so that $\exists$ $\{y_n\} \subseteq X$ s.t.

$$(*) \qquad d(x_n, y_n) < 1 \quad \text{and} \quad f(y_n) \leq \overline{f}(x_n) + \frac{1}{n} , \quad \forall\, n \in \mathbb{N}.$$

Then

$$\inf_{x \in X} \overline{f}(x) \overset{\{x_n\} \text{ is infimizing}}{=} \lim_{n \to +\infty} \overline{f}(x_n) \overset{\{\overline{f}(x_n)\} \text{ is convergent}}{=} \liminf_{n \to +\infty} \overline{f}(x_n)$$

$$\left(\text{As } \tfrac{1}{n} \to 0\right) \to \quad = \liminf_{n \to +\infty} \left[\overline{f}(x_n) + \frac{1}{n} \right]$$

$$\overset{(*)}{\geq} \liminf_{n \to +\infty} f(y_n) \underset{\text{def of inf}}{\geq} \inf_{x \in X} f(x)$$

$\square$

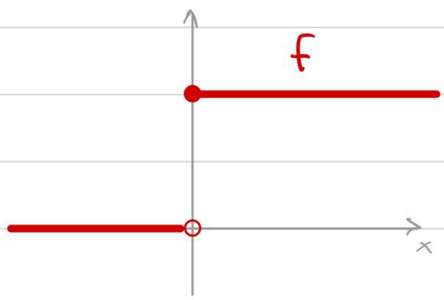
WARNING The statement of PROPOSITION 10.10 only holds on the whole X. In general one has

$$\inf_{x \in A} f(x) > \inf_{x \in A} \overline{f}(x)$$

for $A \subset X$.

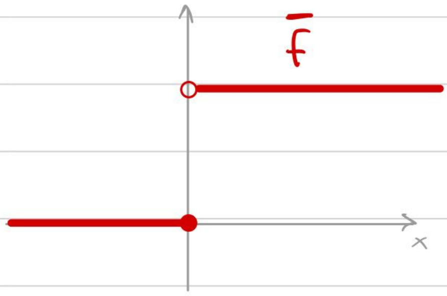
For example consider $X = \mathbb{R}$ and

$$f(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$



Then

$$\bar{f}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$



For $A = [0, +\infty)$ we have

$$\inf_{x \in A} f(x) = 1 \quad , \quad \inf_{x \in A} \bar{f}(x) = 0$$

However the thesis of PROPOSITION 10.10 holds when A is open:

PROPOSITION 10.11 (X, d) metric space, $f : X \to \overline{\mathbb{R}}$ function, $A \subset X$ open. Then

$$\boxed{\inf_{x \in A} f(x) = \inf_{x \in A} \bar{f}(x) = \inf_{x \in A} \hat{f}(x)}$$

Proof. Since $\bar{f} = \hat{f}$ by PROPOSITION 10.9, we only need to show the first equality.

$\boxed{\geq}$ This is clear, since $f \geq \bar{f}$ by PROPOSITION 10.9

$\boxed{\leq}$ Let $\{x_n\}$ be an infimizing sequence for $\bar{f}$ over A, i.e., $\{x_n\} \subseteq A$ and

$$\bar{f}(x_n) \to \inf_{x \in A} \bar{f}(x) .$$

Since A is open, $\forall n \in \mathbb{N}$, $\exists\, r_n > 0$ s.t. $B_{r_n}(x_n) \subset A$.

For all $n \in \mathbb{N}$ apply LEMMA 10.6 with $x = x_n$, $r = r_n$, $\varepsilon = 1/n$, so that $\exists$ $\{y_n\} \subseteq X$ s.t.

$$(*) \qquad d(x_n, y_n) < r_n \qquad \text{and} \qquad f(y_n) \le \bar{f}(x_n) + \frac{1}{n}.$$

The first condition tells us that $y_n \in B_{r_n}(x_n)$, so that $\{y_n\} \subset A$. Then

$\{x_n\}$ is infimizing $\qquad$ $\{\bar{f}(x_n)\}$ is convergent

$$\inf_{x \in A} \bar{f}(x) = \lim_{n \to +\infty} \bar{f}(x_n) = \liminf_{n \to +\infty} \bar{f}(x_n)$$

(As $\frac{1}{n} \to 0$) $\to$
$$= \liminf_{n \to +\infty} \left[\bar{f}(x_n) + \frac{1}{n} \right]$$

$$\overset{(*)}{\ge} \liminf_{n \to +\infty} f(y_n) \ge \inf_{x \in A} f(x)$$

def of inf, since $\{y_n\} \subset A$

$\square$

Now recall the definition of COERCIVE function (DEFINITION 9.5)

DEFINITION X space with notion of convergence. A map $f : X \to \bar{\mathbb{R}}$ is COERCIVE if $\exists$ $K \subset X$ compact s.t.

$$\inf_{x \in X} f(x) = \inf_{x \in K} f(x)$$

For coercive functions on metric space, the following holds:

PROPOSITION 10.12 (X,d) metric space, $f: X \to \overline{\mathbb{R}}$ COERCIVE. Then $\overline{f}$ admits minimum over X and

$$\boxed{\inf_{x \in X} f(x) = \min_{x \in X} \overline{f}(x)}$$

WARNING PROP 10.12 is saying that if f is COERCIVE then the minimum of $\overline{f}$ exists and is equal to the infimum of f.
It is NOT saying that f admits minimum. This is false in general.

Proof As f coercive, $\exists\, K \subset X$ compact s.t.

$$\inf_{x \in X} f(x) = \inf_{x \in K} f(x).$$

By PROPOSITION 10.9 we have that $\overline{f}$ is LSC. As K is compact, from THEOREM 9.4 (DIRECT METHOD) we have that $\overline{f}$ admits minimum on K, i.e.,

$$(*) \qquad \inf_{x \in K} \overline{f}(x) = \min_{x \in K} \overline{f}(x)$$

We CLAIM that $\overline{f}$ admits minimum over X, with

$$(**) \qquad \min_{x \in X} \overline{f}(x) = \min_{x \in K} \overline{f}(x)$$

Let $y \in X$ be arbitrary, and let $\{y_n\} \subset X$ be a RECOVERY SEQUENCE for f at y (which $\exists$ by LEMMA 10.8), i.e.,

$$\overline{f}(y) = \lim_{n \to +\infty} f(y_n)$$

Then

$$\bar{F}(y) \overset{\text{Recovery}}{=} \lim_{n \to +\infty} f(y_n) \overset{\text{Def of inf}}{\geq} \inf_{x \in X} f(x) \overset{\text{Coercivity of } f}{=} \inf_{x \in K} f(x)$$

$$\begin{pmatrix} f \geq \bar{f} \text{ by} \\ \text{PROP } 10.9 \end{pmatrix} \geq \inf_{x \in K} \bar{f}(x) \overset{(*)}{=} \min_{x \in K} \bar{f}(x)$$

Since y was arbitrary, we get

$$\inf_{x \in X} \bar{f}(x) \geq \min_{x \in K} \bar{f}(x)$$

The reverse inequality is obvious, as $K \subset X$. We conclude (**). Therefore

$$\inf_{x \in X} f(x) \underset{\text{PROP 10.10}}{=} \inf_{x \in X} \bar{f}(x) \underset{(**)}{=} \min_{x \in X} \bar{f}(x)$$

□

PROPOSITION 10.13 (Behavior of infimizing sequences)

(X, d) metric space, $f : X \to \bar{\mathbb{R}}$. Suppose that $\{x_n\} \subseteq X$ is s.t.

$$x_n \to x_0 \quad \text{and} \quad f(x_n) \to \inf_{x \in X} f(x)$$ (i.e. $\{x_n\}$ infimizing for f)

Then x_0 is a minimizer for $\bar{f}$, i.e.,

$$\bar{f}(x_0) = \inf_{x \in X} \bar{f}(x)$$

Proof

$$\inf_{x\in X} \bar{f}(x) \le \bar{f}(x_0) \le \liminf_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} f(x_n)$$

def of inf — def of LSC envelope, as $x_n \to x_0$ — as $f(x_n)$ convergent

$$= \inf_{x\in X} f(x) = \inf_{x\in X} \bar{f}(x) \Rightarrow x_0 \in \operatorname{arg\,min}_{x\in X} \bar{f}(x)$$

assumption — PROP 10.10

$\square$

COROLLARY 10.14 (X,d) metric space, $f: X \to \bar{\mathbb{R}}$. Assume that $\exists\, M>0$ and $K \subseteq X$ compact s.t.

$$\tilde{K} := \{x\in X \mid f(x) < M\} \neq \emptyset \quad \text{and} \quad \hat{K} \subseteq K.$$

If $\{x_n\} \subseteq X$ is infimizing for f, i.e.,

$$f(x_n) \to \inf_{x\in X} f(x)$$

then $\exists$ subsequence and $x_0 \in X$ s.t.

$$x_{n_k} \to x_0 \quad \text{and} \quad x_0 \in \operatorname{arg\,min}_{x\in X} \bar{f}(x).$$

Proof Since $\tilde{K} \neq \emptyset$, it means that $I < M$, where $I := \inf\{F(x) \mid x\in X\}$.
As $f(x_n) \to I$, we then conclude that $\exists\, N\in\mathbb{N}$ s.t.

$$x_n \in \tilde{K}, \quad \forall\, n \geq N.$$

As $\tilde{K} \subseteq K$ and K is compact, then $\exists\, x_0 \in K$ and a subsequence s.t. $x_{n_k} \to x_0$. We then conclude from PROP 10.13, since $\{x_{n_k}\}$ is an infimizing sequence for f. $\square$

COMPUTING THE RELAXATION

We will see 2 strategies to compute the relaxation.

PROPOSITION 10.15 (STRATEGY 1)

(X, d) metric space, $F: X \to \overline{\mathbb{R}}$. Suppose that $g: X \to \overline{\mathbb{R}}$ is s.t.

1 (liminf inequality) For all $x \in X$ and $x_n \to x$ it holds

$$g(x) \leq \liminf_{n \to +\infty} F(x_n)$$

2 (limsup inequality) For all $x \in X$, $\exists\, x_n \to x$ s.t.

$$\limsup_{n \to +\infty} F(x_n) \leq g(x)$$

Then $g = \overline{F}$.

NOTE If 1 and 2 hold, then the limsup in 2 is actually a limit.

Proof $\boxed{g \leq \overline{F}}$ Let $x_n \to x$ be arbitrary. By 1 we have

$$g(x) \leq \liminf_{n \to +\infty} F(x_n)$$

Since $\{x_n\}$ is arbitrary, taking the infimum over all sequences $\{x_n\} \subseteq X$ s.t. $x_n \to x$, we get $g \leq \overline{F}$.

$\boxed{\bar{f} \leq g}$ Conversely, let $\{x_n\}$ be the sequence existing by ②. Then

$$\bar{f}(x) \underset{\substack{\uparrow \\ \text{def of } \bar{f} \\ \text{since } x_n \to x}}{\leq} \liminf_{n\to+\infty} f(x_n) \underset{\substack{\uparrow \\ \text{properties of} \\ \text{liminf / limsup}}}{\leq} \limsup_{n\to+\infty} f(x_n) \underset{\substack{\uparrow \\ \text{by ②}}}{\leq} g(x)$$

showing that $\bar{f} \leq g$ and concluding. □

We now look at a second strategy to compute the relaxation.

DEFINITION 10.16 (ENERGY DENSE SUBSETS)

(X,d) metric space, $f: X \to \bar{\mathbb{R}}$. A subset $D \subseteq X$ is ENERGY DENSE WRT f if

$$\forall x \in X, \ \exists \{x_n\} \subseteq D \text{ s.t. } x_n \to x \text{ and } f(x_n) \to f(x).$$

REMARK ① Suppose $f: X \to \mathbb{R}$ is continuous. Then $D \subseteq X$ is Energy Dense WRT to f iff it is Dense.

② $D \subseteq X$ is Energy Dense WRT f iff

$$\{(x, f(x)), \ x \in D\} \subseteq X \times \bar{\mathbb{R}}$$

is dense in $X \times \bar{\mathbb{R}}$.

LEMMA 10.17 (X,d) metric space, $\varphi, \psi : X \to \overline{\mathbb{R}}$. Let $D \subseteq X$.
Suppose that

(i) $\varphi(x) \le \psi(x) \quad \forall x \in D$

(ii) D is Energy Dense WRT ψ

(iii) φ is LSC

Then

$$\varphi(x) \le \psi(x), \quad \forall x \in X.$$

Proof Let $x \in X$. By (ii) there $\exists \{x_n\} \subseteq D$ s.t. $x_n \to x$ and $\psi(x_n) \to \psi(x)$.
Then

$$\underbrace{\varphi(x) \le \liminf_{n\to+\infty} \varphi(x_n)}_{\substack{\varphi \text{ is LSC and} \\ x_n \to x}} \underbrace{\le \liminf_{n\to+\infty} \psi(x_n)}_{\text{By (i), since } \{x_n\} \subseteq D} \underbrace{= \psi(x)}_{\text{As } \psi(x_n) \to \psi(x)}$$

$\square$

PROPOSITION 10.18 (STRATEGY 2)

(X,d) metric space, $f: X \to \overline{\mathbb{R}}$. Suppose that $g: X \to \overline{\mathbb{R}}$ satisfies

1. g is LSC
2. $g(x) \le f(x), \quad \forall x \in X$
3. $\exists D \subseteq X$ Energy Dense WRT g, s.t.

$$\forall x \in D, \ \exists \{x_n\} \subseteq X \text{ s.t. } x_n \to x \text{ and } \limsup_{n\to+\infty} f(x_n) \le g(x)$$

Then $\overline{f} = g$.

<u>Proof</u> $\boxed{g \leq \bar{f}}$ Let $x_n \to x$ be arbitrary. Then

$$\liminf_{n\to+\infty} f(x_n) \overset{②}{\geq} \liminf_{n\to+\infty} g(x_n) \overset{①}{\geq} g(x)$$

Taking the infimum for all $x_n \to x$, we conclude $\bar{f} \geq g$.

$\boxed{\bar{f} \leq g}$ Set $\varphi := \bar{f}$, $\psi := g$. Let us verify the assumptions of LEMMA 10.17:

(i) $\varphi(x) \leq \psi(x) \quad \forall x \in D$ (TRUE because of ③ and definition of $\bar{f}$)

(ii) D is Energy Dense WRT ψ (TRUE: it is assumed in ③)

(iii) φ is LSC (TRUE because $\varphi = \bar{f}$ and $\bar{f}$ is LSC by PROP 10.9)

Therefore by LEMMA 10.17 we have that $\varphi \leq \psi$ on X, i.e. $\bar{f} \leq g$ on X. $\square$