LESSON 12 - 2 JUNE 2021

GENERAL EXISTENCE RESULT IN SOBOLEV

Let $p > 1$, $a < b$, and consider the space

$$X := \{ u \in W^{1,p}(a,b) \mid u(a) = \alpha, \ u(b) = \beta \}$$

Let $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,\xi)$ and define $F : W^{1,p}(a,b) \to \overline{\mathbb{R}}$ by

$$F(u) := \int_a^b L(x, u, \dot{u}) \, dx$$

THEOREM 9.9 Let $p > 1$. Assume L is a Carathéodory function. Suppose that the following conditions hold:

(M1) $\xi \mapsto L(x,s,\xi)$ is CONVEX for a.e. $x \in (a,b)$ and $\forall s \in \mathbb{R}$.

(M2) $\exists \ q \in [1, p)$ and $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$ s.t.

$$L(x,s,\xi) \geq \alpha_1 |\xi|^p + \alpha_2 |s|^q + \alpha_3 , \quad \text{for a.e. } x \in (a,b), \ \forall (s,\xi) \in \mathbb{R} \times \mathbb{R}$$

Set

$$m := \inf \{ F(u) \mid u \in X \} .$$

1. If $m < +\infty$, then $\exists \ u_0 \in X$ which minimizes F over X.

2. If in addition $(s,\xi) \mapsto L(x,s,\xi)$ is STRICTLY CONVEX for a.e. $x \in (a,b)$, then the minimizer is UNIQUE.

REMARK Assumptions (M1)-(M2) in THEOREM 9.9 cannot be weakened.
I will leave some exercises for the Exercise Course to show this claim.

Proof of THEOREM 9.9

Step 1. F is well-defined: Let $u \in W^{1,p}(a,b)$. The map

$$x \mapsto L(x, u(x), \dot{u}(x))$$

is measurable by PROPOSITION 8.2, since L is Carathéodory and $u, \dot{u}$ are measurable. Therefore $x \mapsto L(x, u(x), \dot{u}(x))$ can be integrated and $F(u)$ is well-defined, possibly being infinite.

Step 2. F is weakly LSC:

The proof of weak LSC is very difficult under the assumptions given: see THEOREMS 3.30, 4.1 in B. DACOROGNA - "DIRECT METHODS IN THE CALCULUS OF VARIATIONS", SPRINGER, 2008.

Instead, we prove LSC under much stronger assumptions, just to give an idea of what lies behind it.

Just for this step, assume then that

- $L \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$
- $(s, \xi) \mapsto L(x, s, \xi)$ is convex for every $x \in [a,b]$. (Note that this implies (M1))
- $\exists \; \beta \geq 0$ s.t.

$$|L_s(x,s,\xi)| \, , \, |L_\xi(x,s,\xi)| \leq \beta \left(1 + |s|^{p-1} + |\xi|^{p-1}\right) \, , \quad \forall \, x \in [a,b], \; s, \xi \in \mathbb{R}.$$

We now show that F is weakly LSC, that is,

$$u_n \rightharpoonup u_0 \text{ weakly in } W^{1,p}(a,b) \implies F(u_0) \leq \liminf_{n\to+\infty} F(u_n)$$

Indeed, since L is C^1 and convex WRT (s,ξ), by THEOREM 5.2 we get

$$\begin{aligned} L(x, u_n(x), \dot{u}_n(x)) &\geq L(x, u_0(x), \dot{u}_0(x)) \\ (*) \qquad &+ L_s(x, u_0, \dot{u}_0)(u_n - u_0) \\ &+ L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0) \end{aligned}$$

Notice that

$$(**) \qquad L_s(x, u_0, \dot{u}_0),\ L_\xi(x, u_0, \dot{u}_0) \in L^{p'}(a,b)$$

since

$$\begin{aligned} \int_a^b |L_s(x, u_0, \dot{u}_0)|^{p'} dx &\overset{\text{ASSUMPTION}}{\leq} \beta^{p'} \int_a^b \left(1 + |u_0|^{p-1} + |\dot{u}_0|^{p-1}\right)^{p'} dx \\ \left(\begin{array}{l} p' = \frac{p}{p-1} \text{ and} \\ (a+b)^{p'} \leq 2^{p'-1}(a^{p'}+b^{p'}) \end{array}\right) &\leq \beta^{p'} C \int_a^b |u_0|^p + |\dot{u}_0|^p \, dx \\ &= \beta^{p'} C \|u_0\|^p_{W^{1,p}} < +\infty \end{aligned}$$

The same calculation shows that also $L_\xi(x, u_0, \dot{u}_0) \in L^{p'}(a,b)$.
Then, since $u_n, u_0 \in W^{1,p}(a,b)$, from (**) and Hölder's inequality we get

$$L_s(x, u_0, \dot{u}_0)(u_n - u_0),\ L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0) \in L^1(a,b).$$

Therefore we can integrate (*) to get

$$F(u_n) \geq F(u_0) + \int_a^b L_s(x, u_0, \dot{u}_0)(u_n - u_0)\, dx$$

(***)

$$+ \int_a^b L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0)\, dx$$

Now, $u_n \rightharpoonup u_0$ weakly in $W^{1,p}(a,b)$. In particular $u_n \rightharpoonup u_0$, $\dot{u}_n \rightharpoonup \dot{u}_0$ weakly in $L^p(a,b)$. Since (**) holds, by definition of weak convergence we get

$$\int_a^b L_s(x, u_0, \dot{u}_0)(u_n - u_0)\, dx \,, \quad \int_a^b L_\xi(x, u_0, \dot{u}_0)(\dot{u}_n - \dot{u}_0)\, dx \quad \to 0$$

as $n \to +\infty$. Taking the liminf in (***) yields weak LSC for F.

Step 3. F has COMPACT sublevels :

We are going to prove this part with the original assumptions. So fix $M \in \mathbb{R}$ and let

$$K := \{ u \in X \mid F(u) \leq M \}$$

From (M2) we deduce that $\exists\, M$ such that $K \neq \emptyset$.

By (M2) we have

$$F(u) \overset{(M2)}{\geq} \alpha_1 \|\dot{u}\|_{L^p}^p + \alpha_2 \|u\|_{L^q}^q + \alpha_3 (b-a)$$

(*)

$$\geq \alpha_1 \|\dot{u}\|_{L^p}^p - |\alpha_2| \|u\|_{L^q}^q - |\alpha_3| (b-a)$$

By Hölder inequality we get

Hölder with exponents $p/q > 1$, $(p/q)' = \frac{p}{p-q}$

$$\|u\|_{L^q}^q = \int_a^b |u|^q \, dx \le \left(\int_a^b |u|^p \, dx \right)^{q/p} \left(\int_a^b 1^{p/p-q} \, dx \right)^{\frac{p-q}{p}}$$

$$= \|u\|_{L^p}^q \, (b-a)^{\frac{p-q}{p}}$$

Then from (*)

$$F(u) \overset{(*)}{\geq} \alpha_1 \|\dot{u}\|_{L^p}^p - |\alpha_2| \, \|u\|_{L^q}^q - |\alpha_3| (b-a)$$

(**)

$$\geq \alpha_1 \|\dot{u}\|_{L^p}^p - C_1 \|u\|_{L^p}^q - C_2$$

for some $C_1, C_2 \in \mathbb{R}$. Moreover, if $u \in X$, we have

$$|u(x)| = | u(a) - u(a) + u(x) |$$

(as $u(a) = \alpha$) $\quad \le \alpha + |u(x) - u(a)|$

(THEOREM 7.23, as $p > 1$) $\quad \le \alpha + \|\dot{u}\|_{L^p} \, |x-a|^{1-1/p}$

$$\le \alpha + \|\dot{u}\|_{L^p} \, |b-a|^{1-1/p}$$

and so, integrating the above,

$$\|u\|_{L^p}^q \le c \left\{ 1 + \|\dot{u}\|_{L^p}^q \right\}, \qquad \forall x \in X.$$

Using (**) and the above, we then get some $C_1, C_2 \in \mathbb{R}$ s.t.

$$F(u) \ge \alpha_1 \|\dot{u}\|_{L^p}^p - C_1 \|\dot{u}\|_{L^p}^q - C_2$$

Now let $\{u_n\} \subseteq K$. Then

$$\underbrace{\alpha_1 \|\dot{u}_n\|^p_{L^p} - C_1 \|\dot{u}_n\|^q_{L^p} - C_2}_{\text{Polynomial in } \|\dot{u}_n\|_{L^p}} \overset{\text{Estimate above}}{\leq} F(u_n) \underset{\text{Since } \{u_n\} \subseteq K}{\leq} M$$

As $p > q \geq 1$, we deduce that $\|\dot{u}_n\|_{L^p}$ must be bounded uniformly, i.e.

$$(**) \qquad \|\dot{u}_n\|_{L^p} \leq \tilde{C}, \qquad \forall n \in \mathbb{N}.$$

Since we already proved that

$$\|u\|^q_{L^p} \leq c\,\{1 + \|\dot{u}\|^q_{L^p}\}, \qquad \forall u \in X,$$

from $(**)$ we get

$$\|u_n\|_{W^{1,p}} \leq \tilde{C}, \qquad \forall n \in \mathbb{N}.$$

Recalling that $W^{1,p}$ is a REFLEXIVE BANACH space for $1 < p < +\infty$ (PROPOSITION 7.16) from BANACH - ALAOGLU we conclude the existence of $u_0 \in W^{1,p}(a,b)$ s.t.

$$u_{n_k} \rightharpoonup u_0 \quad \text{weakly in } W^{1,p}(a,b),$$

along some subsequence. By weak LSC of F we also get

$$F(u_0) \leq \liminf_{k \to +\infty} F(u_{n_k}) \underset{\text{As } \{u_n\} \subseteq K}{\leq} M$$

Finally, from the COMPACT embedding $W^{1,p}(a,b) \hookrightarrow C[a,b]$ for $p > 1$ (THEOREM 7.27) we get, by PROPOSITION 7.31,

$$u_{n_k} \to u_0 \quad \text{uniformly in } [a,b].$$

Since $\{u_n\} \subseteq X$, and so $u_n(a) = \alpha$, $u_n(b) = \beta$ $\forall n \in \mathbb{N}$, we conclude

$$u_0(a) = \alpha, \quad u_0(b) = \beta$$

showing that $u_0 \in X$. In total $u_0 \in K$, proving that K is compact.

Step 4. Existence of a minimizer :

So far we have shown that:

- F is weakly LSC in $W^{1,p}(a,b)$
- $\exists\, M \in \mathbb{R}$ s.t.

$$K := \{ u \in X \mid F(u) \leq M \}$$

is non-empty and weakly compact in X.

Thus by the DIRECT METHOD (THEOREM 9.7) we conclude the existence of $\hat{u} \in X$ s.t.

$$F(\hat{u}) = \inf \{ F(u) \mid u \in X \}.$$

Step 5. Uniqueness: Usual stuff: follows as in the proof of THEOREM 5.4, with straightforward adaptations. □