LESSON 11 - 26 MAY 2021

OTHER BOUNDARY CONDITIONS

So far we only dealt with Dirichlet Boundary Conditions. What about other BC? For example one could set the problem in

$$\{ u \in W^{1,p}(a,b) \mid u(a) = \alpha \}$$

This space is well-defined, since Sobolev functions are continuous (THEOREM 7.19).

REMARK Inspecting the proof of THEOREM 8.4 we notice:

- Step 1 - F is well-defined: here we only used the growth assumptions on L to prove that $|F(u)| < +\infty \quad \forall u \in W^{1,p}(a,b)$

- Step 2 - Gâteaux derivative: Here we used (H1) and (H2) separately:

  - Assuming (H1) we proved that F is Gâteaux differentiable at each $u \in W^{1,p}(a,b)$, in every direction $v \in W^{1,p}(a,b)$

  - Assuming (H2) we proved that F is Gâteaux differentiable at each $u \in W^{1,p}(a,b)$, in every direction $v \in C^\infty(a,b)$

- Step 3 - Showing ELE: Here we used the BC

  - Assuming (H1), we chose the variations $v \in W_0^{1,p}(a,b)$. This allowed to deduce (W-ELE), since F was diff. in every direction in $W^{1,p}$

  - Assuming (H2), we chose the variations $v \in C_c^\infty(a,b)$. This allowed to deduce (W'-ELE), since F was diff. in every direction in C^∞

Therefore, we deduce the following general result.

THEOREM 8.5 Let $p \geq 1$, $a < b$. Let $X \subseteq W^{1,p}(a,b)$ be an AFFINE SPACE with reference vector space $V \subseteq W^{1,p}(a,b)$.

Suppose $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies ASSUMPTION 8.3.

Define $F : W^{1,p}(a,b) \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u(x), \dot{u}(x))\, dx .$$

Let $u_0 \in X$ be a minimizer for F over X. Then:

1) If (H1) holds then u_0 satisfies the weak form of ELE

(W-ELE)
$$\boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v}\, dx = 0 , \quad \forall\, v \in V}$$

2) If (H2) holds then u_0 satisfies the weaker form of ELE

(W'-ELE)
$$\boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v}\, dx = 0 , \quad \forall\, v \in C^\infty(a,b) \cap V}$$

3) If in addition $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$ then u_0 satisfies the classical ELE

(ELE)
$$\boxed{\begin{cases} \dfrac{d}{dx}\left[L_\xi(x, u_0, \dot{u}_0) \right] = L_s(x, u_0, \dot{u}_0) , & \forall\, x \in [a,b] \\ \text{BC from the integration by parts of (W'-ELE)} & \end{cases}}$$

SUFFICIENT CONDITIONS FOR MINIMALITY

We now address the generalization of THEOREM 5.4. Let us recall the setting:

Let $p \geq 1$, $a<b$, and let $X \subseteq W^{1,p}(a,b)$ be an affine space over $V \subseteq W^{1,p}(a,b)$.

Let $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,\xi)$ and define $F : W^{1,p}(a,b) \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx$$

The Sobolev version of THEOREM 5.4 (with general BC) is as follows:

THEOREM 8.6 Suppose L satisfies Assumption 8.3.

Assume $u_0 \in X$ solves (W-ELE) or (ELE) in THEOREM 8.5.

1) If

$(s,\xi) \longmapsto L(x,s,\xi)$ is CONVEX for a.e. $x \in (a,b)$

then u_0 is a minimizer for F on X.

2) If

$(s,\xi) \longmapsto L(x,s,\xi)$ is STRICTLY CONVEX for a.e. $x \in (a,b)$

then u_0 is the UNIQUE minimizer for F on X.

(The proof carries out exactly like the one of THEOREM 5.4, with straightforward changes. THEOREM 5.2 can be used because L_s, L_ξ are Carathéodory. Hence L is C^1 WRT (s,ξ), for a.e. $x \in (a,b)$ fixed).

9. DIRECT METHOD

GOAL The Direct Method is used to prove existence of minimizers

We state a very general version of the direct method, for functionals $F: X \to \mathbb{R}$ with X space with a notion of convergence.

DEFINITION 9.1 Let X be a set, and $Seq(X)$ the set of all sequences in X:

$$Seq(X) := \{ f : \mathbb{N} \to X \}.$$

A **NOTION of CONVERGENCE** on X is a subset N of

$$Seq(X) \times X$$

contains sequences   contains limits

NOTATION Thus a notion of convergence is a list of sequences with corresponding limit. Therefore, whenever we say that $\{x_n\} \subseteq X$ converges to $x_0 \in X$, in symbols $x_n \to x_0$, we mean that $(\{x_n\}, x_0) \in N$ notion of convergence.

EXAMPLES

- X = topological space. A notion of convergence is for example the list of all sequences $\{x_n\} \subseteq X$ s.t. $x_n \to x_0$ WRT to τ, for some $x_0 \in X$
- The above example contains all the well-known cases: Metric spaces, Normed spaces with weak or strong convergence, Hilbert spaces, $\mathbb{R}^d$.

DEFINITION 9.2 X space with notion of convergence. We say that $K \subseteq X$ is (sequentially) compact if every sequence $\{x_n\} \subseteq K$ admits a subsequence such that $x_{n_k} \to x_0$ with $x_0 \in K$.

DEFINITION 9.3 X space with notion of convergence. A function $f: X \to \mathbb{R}$ is

- **CONTINUOUS** if for all sequences $x_n \to x_0$ we have

$$f(x_n) \to f(x_0).$$

- **LOWER SEMICONTINUOUS (LSC)** if for all $x_n \to x_0$ we have

$$f(x_0) \leq \liminf_{n \to +\infty} f(x_n).$$

THEOREM 9.4 (DIRECT METHOD)

X space with notion of convergence, $f: X \to \mathbb{R}$. Assume that

(i) X is compact

(ii) f is LSC

WRT the SAME notion of convergence

Then the problem

$$I := \inf \{ f(x) \mid x \in X \}$$

admits solution: i.e., $\exists \hat{x} \in X$ s.t. $f(\hat{x}) = I$.

Proof Exactly like the Weierstraß Theorem of Analysis 1.

By the properties of the infimum $\exists$ infimizing sequence $\{y_n\} \subseteq f(X)$ s.t.

$$y_n \to I.$$

Note that, a priori, $I \in [-\infty, +\infty]$.

By def. of image $\exists \{x_n\} \subseteq X$ s.t. $y_n = f(x_n)$, $\forall n \in \mathbb{N}$. Thus

$$f(x_n) \to I.$$

As X is compact, there $\exists$ a subsequence s.t. $x_{n_k} \to \hat{x}$ for some $\hat{x} \in X$. Then

$$\underset{\substack{\uparrow \\ I \text{ is inf}}}{I \leq} f(\hat{x}) \underset{\substack{\uparrow \\ f \text{ is LSC} \\ \text{and } x_{n_k} \to \hat{x}}}{\leq} \liminf_{k \to +\infty} f(x_{n_k}) \underset{\substack{\uparrow \\ f(x_n) \text{ is convergent}}}{=} \lim_{n \to +\infty} f(x_n) = I$$

Thus $f(\hat{x}) = I$, concluding that $\hat{x}$ is a minimizer and that I is finite. $\square$

REMARK The direct method is deceivingly simple. The highly non-trivial task is finding a notion of convergence on X s.t. (i)-(ii) hold. Note that:

- If we have many convergent sequences then (i) is easy and (ii) hard
- If we have few convergent sequences then (i) is hard and (ii) easy

Therefore (i) and (ii) are in competition, and and finding a notion of convergence s.t. both hold is delicate.

Let us now see some variants of the direct method.

DEFINITION 9.5 X space with notion of convergence. A function $f : X \to \mathbb{R}$ is **COERCIVE** if $\exists K \subseteq X$ compact s.t.

$$\inf \{ f(x) \mid x \in K \} = \inf \{ f(x) \mid x \in X \}$$

EXAMPLE $X = \mathbb{R}$, $f(x) = x^2$. Then f is coercive (i.e. $K = [-1, 1]$)

THEOREM 9.6 X space with notion of convergence, $f: X \to \mathbb{R}$ s.t.

(i) f is COERCIVE

(ii) f is LSC

Then

$$I := \inf \{ f(x) \mid x \in X \}$$

admits solution.

Proof As f is coercive then by definition $\exists\, K \subseteq X$ compact s.t.

$$(*) \quad I = \inf \{ f(x) \mid x \in K \}$$

As K is compact and f is LSC on K (as f is LSC on X), we can apply THEOREM 9.4 to obtain $\hat{x} \in K$ s.t. $f(\hat{x}) = \inf \{ f(x) \mid x \in K \}$. Thus, by $(*)$, $f(\hat{x}) = I$ and we conclude. $\square$

THEOREM 9.7 X space with notion of convergence, $f: X \to \mathbb{R}$ s.t.

(i) $\exists\, M \geq 0$, $\exists\, K \subseteq X$ compact s.t. $\{x \in X \mid f(x) \leq M\} \neq \emptyset$ and

$$\{x \in X \mid f(x) \leq M\} \subseteq K$$

(ii) f is LSC

Then

$$I := \inf \{ f(x) \mid x \in X \}$$

admits solution $\hat{x} \in X$ s.t. $f(\hat{x}) \leq M$.

Proof We want to show that $\tilde{K} := \{x \in X \mid f(x) \leq M\}$ is compact. So let $\{x_n\} \subseteq \tilde{K}$. As $\tilde{K} \subseteq K$ and K is compact, there $\exists$ a subsequence and $x_0 \in K$ s.t.

$$x_{n_k} \to x_0 .$$

Since $\{x_{n_k}\} \subseteq \tilde{K}$ and f is LSC, we get

$$f(x_0) \leq \liminf_{n \to +\infty} f(x_{n_k}) \leq M$$

f LSC and $x_{n_k} \to x_0$

As $\{x_{n_k}\} \subseteq \tilde{K}$

proving that $x_0 \in \tilde{K}$ and so that $\tilde{K}$ is compact. We now have two cases:

- $I = M$: then by def. of $\tilde{K}$ and of infimum

$$\tilde{K} = \{x \in X \mid f(x) \leq I\} = \{x \in X \mid f(x) = I\}.$$

Thus $\tilde{K}$ is exactly set of minimizers. Since $\tilde{K} \neq \emptyset$ by assumption, we conclude.

- $I < M$: Let $\{x_n\} \subseteq X$ be an infimizing sequence, i.e., such that

$$f(x_n) \to I .$$

Since $I < M$, we conclude that $\{x_n\} \subseteq \tilde{K}$. (upon discarding a finite number of indices). As $\tilde{K}$ is compact, $\exists$ a subsequence and $\hat{x} \in \tilde{K}$ s.t. $x_{n_k} \to \hat{x}$. Now

$$I \leq f(\hat{x}) \leq \liminf_{k \to +\infty} f(x_{n_k}) = \lim_{n \to +\infty} f(x_n) = I ,$$

def of inf

$x_{n_k} \to \hat{x}$ and f is LSC

as $f(x_n)$ is convergent

Thus $f(\hat{x}) = I$ and we conclude that $\hat{x}$ is a minimizer. $\square$

DIRECT METHOD - ACTION PLAN

Given $F : X \to \mathbb{R}$, the minimization problem

$$(P) \quad \inf\{F(x) \mid x \in X\}$$

can be studied in the following way:

① **WEAK FORMULATION**: Extend F to a functional $\hat{F} : \hat{X} \to \mathbb{R}$ with $X \subseteq \hat{X}$, i.e., to a larger space
(Typically $\hat{X}$ will by a SOBOLEV SPACE, rather than the usual C^1, C^∞ or C^1_{pw})

② **COMPACTNESS**: Prove that the sublevel sets of $\hat{F}$ are compact WRT some appropriate notion of convergence on $\hat{X}$

③ **LOWER SEMICONTINUITY**: Prove that $\hat{F}$ is LSC WRT the same notion of convergence of point ②.

④ **REGULARITY**: At this point one can apply THEOREM 9.7 and conclude the $\exists$ of a solution $\bar{x} \in \hat{X}$ to

$$\inf\{\hat{F}(x) \mid x \in \hat{X}\}$$

The last step consists in showing that

$$\bar{x} \text{ is more regular, i.e., } \bar{x} \in X$$

Note that, as $\hat{F} = F$ on X, this immediately implies that $\bar{x}$ solves the original minimization problem (P)

EXAMPLE 9.8 Set $X = \{ u \in C^1[0,1] \mid u(0)=0, \ u(1)=1 \}$ and

$$F(u) := \int_0^1 \dot{u}^2 + \sin(u^5)\, dx , \qquad u \in X .$$

Note that the Lagrangian appearing in F is non-linear. Thus the associated ELE is hard (maybe impossible) to solve explicitly.

We then resort to our ACTION PLAN for the DIRECT METHOD:

(1) **WEAK FORMULATION**: We extend F to the larger space

$$\hat{X} := \{ u \in H^1(0,1) \mid u(0)=0, \quad u(1)=1 \} .$$

Note that $\hat{X}$ is well-defined, since H^1 functions are continuous by THEOREM 7.19. Therefore the Dirichlet Boundary conditions appearing in $\hat{X}$ make sense.

The extension of F to $\hat{X}$ is trivially defined by

$$\hat{F}(u) := \int_0^1 \underbrace{\dot{u}}_{\text{WEAK DERIVATIVE}}{}^2 + \sin(u^5)\, dx , \qquad \forall u \in H^1(0,1) ,$$

Note that $\hat{F}$ is well-defined, since

- $\dot{u} \in L^2(0,1)$ as $u \in H^1(0,1)$
- $\sin(u^5) \in L^1(0,1)$ as $H^1(0,1) \hookrightarrow L^\infty(0,1)$ by the SOBOLEV EMBEDDING THEOREM 7.27 (or, more simply, because $|\sin x| \leq 1$)

Moreover $\hat{F} = F$ on X, since if $u \in C^1[0,1]$, then its weak derivative $\dot{u}$ coincides a.e. with the classical derivative.

② COMPACTNESS :

We need to show that there $\exists\ M>0$ s.t. the sublevel

$$K := \{ u \in \hat{X} \mid \hat{F}(u) \le M \}$$

is non-empty, and compact WRT some notion of convergence on $H^1(0,1)$. Clearly we can choose $M := F(u)$ with $u(x) := x$, so that $K \ne \emptyset$.

As notion of convergence we take the weak convergence on H^1. We have to show that K is compact. Hence assume that $\{u_n\} \subseteq K$, that is,

$$\{u_n\} \subseteq \hat{X} \quad \text{and} \quad \hat{F}(u_n) \le M, \quad \forall\ n \in \mathbb{N}.$$

As $|\sin x| \le 1$, we get

$$\int_0^1 \dot{u}_n^2\, dx - 1 \le \int_0^1 \dot{u}_n^2 + \sin(u^5)\, dx = \hat{F}(u_n) \le M \quad \Rightarrow \quad \boxed{\|\dot{u}_n\|_{L^2} \le \sqrt{M+1}}$$

Thus $\{\dot{u}_n\}$ is uniformly bounded in $L^2(0,1)$. Since $L^2(0,1)$ is Hilbert separable, by Banach-Alaoglu Theorem we conclude that there $\exists$ a subseq. and $\hat{v} \in L^2(0,1)$ s.t.

$$\dot{u}_n \rightharpoonup \hat{v} \quad \text{weakly in } L^2(0,1).$$

Moreover by the Hölder estimate of THEOREM 7.23 (with $p=2$) we get

$$\begin{aligned} |u_n(x) - u_n(y)| &\le \|u_n'\|_{L^2}\, |x-y|^{1/2} \\ &\le \sqrt{M+1}\, |x-y|^{1/2}, \qquad \forall x,y \in [0,1], \end{aligned} \qquad (*)$$

showing that $\{u_n\}$ is EQUI-CONTINUOUS.

Using the boundary condition $u_n(0)=0$ we also get

$$|u_n(x)| = |u_n(x) - u_n(0)| \overset{*}{\leq} \sqrt{M+1} \quad , \qquad \forall x \in [0,1],$$

showing that $\{u_n\}$ is UNIFORMLY BOUNDED in $C[0,1]$. Therefore we can apply ASCOLI-ARZELA' THEOREM 7.28 to conclude that $\overline{\{u_n\}}$ is COMPACT in $C[0,1]$. Then $\exists$ a subsequence and $\hat{u} \in C[0,1]$ s.t.

$$u_{n_k} \to \hat{u} \quad \text{uniformly in } [0,1].$$

In particular $u_{n_k} \to \hat{u}$ strongly in $L^2(0,1)$ and so

$$u_{n_k} \rightharpoonup \hat{u} \quad \text{weakly in } L^2(0,1).$$

Recalling that $\dot{u}_{n_k} \rightharpoonup \hat{v}$ weakly in $L^2(0,1)$, by REMARK 7.18 we get that

$$u_{n_k} \rightharpoonup \hat{u} \text{ weakly in } H^1(0,1), \quad \text{with } \hat{u}' = \hat{v} \text{ in the weak sense.}$$

In particular $\hat{u} \in H^1(0,1)$, and $\hat{u}(0)=0$, $\hat{u}(1)=1$ by the uniform convergence. Thus $\hat{u} \in \hat{X}$. As norms are weakly lower semicontinuous, we get that

$$\int_0^1 (\hat{u}')^2 \, dx \leq \liminf_{k \to +\infty} \int_0^1 u_{n_k}' \, dx$$

Also, since $u_{n_k} \to \hat{u}$ uniformly,

$$\lim_{k \to +\infty} \int_0^1 \sin(u_n^5) \, dx = \int_0^1 \sin(\hat{u}^5) \, dx$$

Therefore

$$\hat{F}(\hat{u}) = \int_0^1 (\hat{u}')^2 + \sin(\hat{u}^5) \, dx \leq \liminf_{k \to +\infty} \int_0^1 \dot{u}_n^2 + \sin(u_n^5) \, dx = \liminf_{k \to +\infty} \hat{F}(u_n) \leq M$$

showing that $\hat{F}(\hat{u}) \leq M$. Thus $\hat{u} \in K$, proving that K is weakly compact.

③ **LOWER SEMICONTINUITY :**

(Here we could conclude with the same arguments of point ②. But it is instructive to make a separate argument.)

We need to prove that $\hat{F}$ is lower semicontinuous WRT the weak convergence of H^1, that is,

$$(S) \quad u_n \to u \text{ weakly in } H^1(0,1) \quad \Rightarrow \quad \hat{F}(u) \leq \liminf_{n \to +\infty} \hat{F}(u_n)$$

From the SOBOLEV EMBEDDING THEOREM 7.27 we have $H^1(0,1) \hookrightarrow C[0,1]$ compactly. Now recall that COMPACT OPERATORS transform weakly convergent sequences into strongly convergent sequences (PROPOSITION 7.31). Therefore

$$u_n \to u \text{ weakly in } H^1(0,1) \quad \Rightarrow \quad u_n \to u \text{ uniformly in } [0,1]$$

From the weak lower semicontinuity of the norm, we obtain

$$\int_0^1 \dot{u}^2 \, dx \leq \liminf_{n \to +\infty} \int_0^1 \dot{u}_n^2 \, dx \qquad (\text{since } \dot{u}_n \to \dot{u} \text{ weakly in } L^2(0,1))$$

Moreover, as $u_n \to u$ uniformly, we also have

$$\int_0^1 \sin(u^5) \, dx = \lim_{n \to +\infty} \int_0^1 \sin(u_n^5) \, dx.$$

Thus $\hat{F}(u) \leq \liminf_{n \to +\infty} \hat{F}(u_n)$, and (S) is proven.

Therefore, by THEOREM 9.7 we conclude the existence of $\bar{u} \in \hat{X}$ s.t.

$$\boxed{\hat{F}(\bar{u}) = \inf \{ \hat{F}(u) \mid u \in \hat{X} \}}$$

④ **REGULARITY**: We wish to show that $\bar{u} \in \hat{X}$ actually belongs to X, so that it automatically solves the original problem

$$F(\bar{u}) = \inf \{ F(u) \mid u \in X \}$$

CLAIM All minimizers of $\hat{F}$ in $\hat{X}$ belong to $C^\infty(0,1)$.

HOW TO PROVE IT
- 4.1 : WRITE THE WEAK ELE FOR $\hat{F}$
- 4.2 : SHOW THAT u_0 IS CONTINUOUS (where u_0 is minimizer)
- 4.3 : BOOTSTRAP ARGUMENT

Proof of Claim Let $u_0 \in \hat{X}$ be a minimizer for $\hat{F}$. We want to apply THEOREM 8.4 (with $p=2$) to derive the ELE.

Since u_0 is not regular for now, we can only hope that either the WEAK ELE, or worse the VERY WEAK ELE, hold. So let us check ASSUMPTION 8.3.
In our case the Lagrangian is

$$L(x,s,\xi) = \xi^2 + \sin(s^5)$$

- L is C^∞, therefore L, L_s, L_ξ are Carathéodory functions

- We check (H1): we need to show that $\forall R > 0$, $\exists\, \alpha_1 \in L^1(a,b)$, $\alpha_2 \in L^{p'}(a,b)$, $\beta = \beta(R)$ s.t.

(1) $|L(x,s,\xi)| \leq \alpha_1(x) + \beta |\xi|^p$

(2) $|L_s(x,s,\xi)| \leq \alpha_1(x) + \beta |\xi|^p$, $\forall x \in (a,b)$, $|s| \leq R$, $\xi \in \mathbb{R}$

(3) $|L_\xi(x,s,\xi)| \leq \alpha_2(x) + \beta |\xi|^{p-1}$

Notice that in our case $a=0$, $b=1$ and $p=2$, so that $p'=2$.
Let us check (1):

$$|L(x,s,\xi)| = |\xi^2 + \sin(s^5)| \leq 1+\xi^2 \quad , \quad \forall x\in(0,1),\ s,\xi\in\mathbb{R}$$

Therefore it looks like we can choose $\alpha_1 \equiv 1$ independent on x (since the RHS does not depend on x) and, $\beta \equiv 1$ independent on R (since the estimate holds for all $s\in\mathbb{R}$).

Let us see if this choice of α_1 and β works for (2):

$$|L_s(x,s,\xi)| = |5s^4\cos(s^5)| \leq 5|s|^4$$

This estimate can resemble (2) only if we assume $|s|\leq R$, in which case we get

$$|L_s(x,s,\xi)| \leq 5R^4 \quad , \quad \forall x\in(0,1),\ |s|\leq R,\ \xi\in\mathbb{R}.$$

This is saying that we should have $\alpha_1 \equiv 5R^4$ and $\beta \equiv 0$.

Let us look into (3):

$$|L_\xi(x,s,\xi)| = 2|\xi| \quad , \quad \forall x\in(0,1),\ s\in\mathbb{R},\ \xi\in\mathbb{R}$$

Therefore (3) is satisfied for $\alpha_2 \equiv 0$ and $\beta \equiv 2$.

Then, it is immediate to check that L satisfies (1), (2), (3) for

$$\alpha_1(x) \equiv \max\{1, 5R^4\} \quad , \quad \alpha_2(x) \equiv 0 \quad , \quad \beta = 2$$

Since $\alpha_1, \alpha_2 \in L^1(0,1)$, we get that (H1) holds.

Therefore L satisfies ASSUMPTION 8.3, and since $u_0 \in X$ is a minimizer of $\hat{F}$ over $\hat{X}$, by THEOREM 8.4 we get that u_0 satisfies the WEAK ELE

$$\int_0^1 L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\, \dot{v} \; dx = 0, \quad \forall\, v \in W_0^{1,2}(0,1),$$

which in our case reads

$$\int_0^1 5\, u_0^4 \cos(u_0^5)\, v + 2\, \dot{u}_0\, \dot{v} \; dx = 0, \qquad \forall\, v \in W_0^{1,2}(0,1).$$

Rearranging we get

$$\text{(W-ELE)} \quad \int_0^1 \underbrace{2\,\dot{u}_0}_{f}\, \underbrace{\dot{v}}_{\dot{\varphi}} \, dx = - \int_0^1 \underbrace{5\, u_0^4 \cos(u_0^5)}_{g}\, \underbrace{v}_{\varphi} \, dx, \qquad \forall\, v \in W_0^{1,2}(0,1).$$

Recalling that $C_c^1(0,1) \subseteq W_0^{1,2}(0,1)$, (W-ELE) is saying that $f := 2\dot{u}_0$ is weakly differentiable, with weak derivative given by $g := 5\, u_0^4 \cos(u_0^5)$, that is

$$(*) \qquad (2\,\dot{u}_0)' = 5\, u_0^4 \cos(u_0^5) \qquad \text{weakly}$$

We use $(*)$ to prove regularity of u_0. Note that

$$\int_0^1 |g|^2 dx \underset{|\cos x| \le 1}{\le} 25 \int_0^1 |u_0|^8 \, dx$$

and the RHS is finite, since $u_0 \in W^{1,2}(0,1)$ and $W^{1,2}(0,1) \hookrightarrow L^\infty(0,1)$ continuously by THEOREM 7.27.

This shows $g \in L^2(0,1)$. But then

$$\dot{f} = g \text{ weakly}, \; g \in L^2 \Rightarrow f \in W^{1,2}(0,1) \Rightarrow \dot{u}_0 \in W^{1,2}(0,1)$$

$f = 2\dot{u}_0$

But

$$\dot{u}_0 \in W^{1,2}(0,1) \overset{\text{THM 7.19}}{\Rightarrow} \dot{u}_0 \in C[0,1]$$

(Note: here $\dot{u}_0$ is still a weak derivative)

Now, by PROPOSITION 7.22 we have that, as $u_0 \in W^{1,2}$ and the weak derivative $\dot{u}_0$ is continuous, then

$$\boxed{u_0 \in C^1[0,1]}$$

Then

$$u_0 \text{ is } C^1 \Rightarrow g = 5u_0^4 \cos(u_0^5) \text{ is } C^1 \Rightarrow g \in C^0$$

(As $(2\dot{u}_0)' = g$ weakly) $\Rightarrow$ $2\dot{u}_0$ has continuous weak derivative

$$(\text{PROP 7.22}) \Rightarrow \dot{u}_0 \in C^1 \Rightarrow u_0 \in C^2$$

this is true because $\dot{u}_0$ is a classical derivative

Now that we proved $u_0 \in C^2$, we can employ the BOOTSTRAP argument.

BOOTSTRAP: Since now we know $u_0 \in C^2$, the relationship

$$(2\dot{u}_0)' = 5u_0^4 \cos(u_0^5) \qquad \text{weakly}$$

holds in the classical sense (the weak derivative of a diff. function is just the classical derivative), i.e.,

$$(**) \qquad 2\ddot{u}_0 = 5u_0^4 \cos(u_0^5) \quad , \quad \forall x \in [0,1].$$

Then, as $u_0 \in C^2$, the RHS of (**) belongs to C^2, and so

$$\ddot{u}_0 \in C^2 \Rightarrow u_0 \in C^4$$

Again, as $u_0 \in C^4$, the RHS of (**) belongs to C^4, and so

$$\ddot{u}_0 \in C^4 \Longrightarrow u_0 \in C^6$$

Proceeding with the botstrap argument we conclude that

$$\ddot{u}_0 \in C^k \Longrightarrow u_0 \in C^{k+2}$$

and therefore $u_0 \in C^\infty(0,1)$. $\square$

To summarize, in EXAMPLE 9.8 we proved the following:

PROPOSITION Let $X := \{ u \in C^1[0,1] \mid u(0)=0, \ u(1)=1 \}$ and

$$F(u) := \int_0^1 \dot{u}^2 + \sin(u^5)\, dx .$$

Then F admits a minimizer $\bar{u} \in X \cap C^\infty(0,1)$.

NOTE: The remarkable feature of this ACTION PLAN for the DIRECT METHOD is that we never tried to solve the ELE equation, but just use abstract arguments to prove $\exists$ of a minimizer, and then the structure of ELE to recover Regularity.