LESSON 10 - 19 MAY 2021

HIGHER ORDER SOBOLEV SPACES

We can of course generalize the definition of Sobolev function to higher order derivatives:

DEFINITION Let $k \geq 2$ be an integer, $1 \leq p \leq +\infty$. Let $I \subseteq \mathbb{R}$ be an open set. We define

$$W^{k,p}(I) := \{ u \in W^{k-1,p}(I) \mid u' \in W^{k-1,p}(I) \}.$$

For $p=2$ we set

$$H^k(I) := W^{k,2}(I).$$

REMARK $u \in W^{k,p}(I)$ iff $\exists\ g_1, \dots, g_k \in L^p(I)$ s.t.

$$\int_I u \varphi^{(j)} \, dx = (-1)^j \int_I g_j \varphi \, dx \ , \quad \forall \varphi \in C_c^\infty(I) \ , \ \forall j = 1, \dots, k \ ,$$

i.e. u admits weak derivatives up to order k. (easy check)

NOTATION In view of the above remark, and due to the uniqueness of weak derivatives, if $u \in W^{k,p}(I)$ we denote by

$$u^{(j)} := g_j \ , \quad j = 1, \dots, k$$

the j-th weak derivative.

PROPOSITION 7.33 Let $I \subseteq \mathbb{R}$ be open, $k \geq 2$ be an integer, $1 \leq p \leq +\infty$. Then, the space $W^{k,p}(I)$ is Banach with the norm

$$\boxed{\|u\|_{W^{k,p}} := \|u\|_{L^p} + \sum_{j=1}^{k} \|u^{(j)}\|_{L^p}}$$

Moreover $H^k(I)$ is Hilbert with the inner product

$$\boxed{\langle u, v \rangle_{H^k} := \langle u, v \rangle_{L^2} + \sum_{j=1}^{k} \langle u^{(j)}, v^{(j)} \rangle_{L^2}}$$

(The proof is obtained following the lines of the proof of PROPOSITION 7.16)

REMARK $I \subseteq \mathbb{R}$ open, $k \geq 2$, $1 \leq p \leq +\infty$. Then $W^{k,p}(I) \subseteq C^{k-1}(\bar{I})$.

(Proof is consequence of THEOREM 7.19. For example, for $k=2$ we have that if $u \in W^{2,p}(I)$, then by definition $u' \in W^{1,p}(I)$.
As $W^{1,p}(I) \subset C(\bar{I})$ by THM 7.19, we get that $u' \in C(\bar{I})$. Therefore

$$u \in W^{1,p}(I), \quad u' \in C(I)$$

and thus, by PROPOSITION 7.22 we get $u \in C^1(\bar{I})$, concluding that

$$W^{2,p}(I) \subset C^1(\bar{I}).$$

Similarly, one can conclude the other cases.).

THE SPACE $W_0^{1,p}$

When dealing with Dirichlet type boundary conditions, it is useful to introduce the space $W_0^{1,p}$, which will be the space of functions $u \in W^{1,p}$ s.t. $u = 0$ on ∂I.

DEFINITION Let $I \subseteq \mathbb{R}$ be open, $1 \le p < +\infty$. The space $W_0^{1,p}(I)$ is defined as the CLOSURE of $C_c^1(I)$ in $W^{1,p}(I)$. We denote

$$H_0^1(I) := W_0^{1,2}(I).$$

The space $W_0^{1,p}(I)$ is equipped with the norm of $W^{1,p}(I)$, while $H_0^1(I)$ is equipped with the inner product of $H^1(I)$.

REMARK
- $W_0^{1,p}$ is a SEPARABLE BANACH space
- $W_0^{1,p}$ is REFLEXIVE for $1 < p < +\infty$
- H_0^1 is a SEPARABLE HILBERT space

(These follow from PROPOSITION 7.16 and the fact that $W_0^{1,p}$ is closed by definition)

REMARK By THEOREM 7.24 we know that $C_c^1(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$. Therefore

$$W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R}).$$

THEOREM 7.34 Let $I \subseteq \mathbb{R}$ be open, $1 \le p < +\infty$. They are equivalent:

(a) $u \in W_0^{1,p}(I)$

(b) $u = 0$ on ∂I

We only prove the easy implication of THEOREM 7.34, that is, (a) $\Rightarrow$ (b).

Proof

(a) $\Rightarrow$ (b): By definition, if $u \in W_0^{1,p}(I)$ there $\exists \{u_n\} \subseteq C_c^1(I)$ s.t. $u_n \to u$ strongly in $W^{1,p}(I)$. By the embedding $W^{1,p}(I) \hookrightarrow C(\bar{I})$ (THEOREMS 7.19 and 7.27) we get that $u_n \to u$ uniformly in $\bar{I}$. As $u_n = 0$ on ∂I we then conclude $u = 0$ on ∂I.

(b) $\Rightarrow$ (a): See THEOREM 8.12 in BREZIS – "Functional Analysis, Sobolev Spaces and PDE", SPRINGER 2011. $\square$

POINCARÉ INEQUALITIES

THEOREM 7.35 (POINCARÉ INEQUALITY)

Let $I=(a,b)$ be bounded, $1 \le p < +\infty$. There $\exists$ $C>0$ (depending only on $|I|$) s.t.

$$\|u\|_{W^{1,p}(I)} \le C \|u'\|_{L^p(I)}, \quad \forall u \in W_0^{1,p}(I).$$

In particular $\|u\|_{W^{1,p}(I)}$ and $\|u'\|_{L^p(I)}$ are equivalent norms on $W_0^{1,p}(I)$.

We give two proofs: the first one is more direct, while the second one is more abstract, but useful for proving generalizations.

WARNING The Poincaré Inequality does not hold in $W^{1,p}(a,b)$ (think of constants)

Proof 1 Let $u \in W_0^{1,p}(a,b)$. As $u(a)=0$ by THEOREM 7.34, we get

$$|u(x)| = |u(x)-u(a)|$$

(Here use THEOREM 7.19) $\longrightarrow$ $$= \left| \int_a^x u'(x)\,dx \right| \le \|u'\|_{L^1(a,b)}$$

Therefore $\|u\|_{L^\infty(a,b)} \le \|u'\|_{L^1(a,b)}$. Then

$$(*) \quad \|u\|^p_{L^p(a,b)} = \int_a^b |u|^p\,dx \le (b-a)\|u\|^p_{L^\infty(a,b)} \le (b-a)\|u'\|^p_{L^1(a,b)}$$

By Hölder's inequality we get

$$(**) \quad \|u'\|_{L^1(a,b)} \le \left(\int_a^b |u'|^p\,dx\right)^{1/p} \left(\int_a^b 1^{p'}\,dx\right)^{1/p'}$$

$$= \|u'\|_{L^p(a,b)}\,(b-a)^{1/p'}$$

Thus, by (*),

$$\|u\|_{L^p(a,b)} \overset{(*)}{\leq} (b-a)^{1/p} \|u'\|_{L^1(a,b)}$$

Since $\frac{1}{p}+\frac{1}{p'}=1$

$$\overset{(**)}{\leq} (b-a)^{1/p+1/p'} \|u'\|_{L^p(a,b)} = (b-a)\|u'\|_{L^p(a,b)}$$

Now

$$\|u\|_{W^{1,p}(a,b)} = \|u\|_{L^p(a,b)} + \|u'\|_{L^p(a,b)} \leq (b-a+1)\|u'\|_{L^p(a,b)}$$

therefore we conclude setting $C := b-a+1 = |I|+1$. □

Proof 2 Assume by contradiction that the inequality does not hold. Then we can find a sequence $\{u_n\} \subseteq W_0^{1,p}(a,b)$ s.t.

$$(*) \quad \|u_n\|_{L^p} \geq n \|u_n'\|_{L^p}, \quad \forall n \in \mathbb{N}.$$

As the norm is homogeneous, up to rescaling u_n by $\|u_n\|_{L^p}$, we can assume that $\|u_n\|_{L^p}=1$, $\forall n \in \mathbb{N}$. Then, from (*), we get

$$(**) \quad \|u_n\|_{L^p}=1, \quad \|u_n'\|_{L^p} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

In particular $\{u_n\}$ is bounded in $W^{1,p}(a,b)$. By the SOBOLEV EMBEDDING THM 7.27 (point (c)) we know that $W^{1,p}(a,b) \hookrightarrow L^p(a,b)$ compactly. Thus $\overline{\{u_n\}}$ is compact in $L^p(a,b)$. In particular $\{u_n\}$ admits a subsequence s.t.

$$u_{n_k} \to u \text{ strongly in } L^p(a,b).$$

Moreover, from (**) we know that $\|u_{n_k}'\|_{L^p} \leq \frac{1}{n_k}$, $\forall k \in \mathbb{N}$. Therefore

$$u_{n_k}' \to 0 \text{ strongly in } L^p(a,b).$$

Thus, from REMARK 7.17 we conclude that $u_{n_k} \to u$ strongly in $W^{1,p}(a,b)$, with $u' = 0$ in the weak sense.

Therefore, by definition of weak derivative, we get

$$\int_a^b u \varphi' \, dx = - \int_a^b u' \varphi \, dx \overset{u'=0}{=} 0 \quad , \quad \forall \varphi \in C_c^1(a,b)$$

and the DBR LEMMA 7.13 implies that $u = c$ a.e. on (a,b), for some $c \in \mathbb{R}$.

Now recall that $W_0^{1,p}(a,b)$ is closed by definition, therefore, as $u_{n_k} \to u$ in $W^{1,p}(a,b)$, and $\{u_n\} \subseteq W_0^{1,p}(a,b)$, we get that $u \in W_0^{1,p}(a,b)$.

By THEOREM 7.34 we then have $u(a) = u(b) = 0$. Since $u = c$, this implies $c = 0$ and

$$u \equiv 0 .$$

However, taking the limit as $k \to +\infty$ in the first condition in (**) gives

$$\|u\|_{L^p} = 1 ,$$

which is a contradiction, as $u \equiv 0$. $\square$

When dealing with BC which are more general than homogeneous Dirichlet BC, the above Poincaré inequality is useless.

Therefore we look for a more general version. In order to do that, notice that the Poincaré Inequality

$$\|u\|_{W^{1,p}(I)} \le C \|u'\|_{L^p(I)} \quad , \quad \forall u \in W_0^{1,p}(I)$$

holds because non-zero constant functions do not belong to $W_0^{1,p}$.

This simple observation motivates the following generalization of THEOREM 7.35.

THEOREM 7.36 (GENERALIZED POINCARÉ INEQUALITY)

Let $I=(a,b)$ be bounded, $1 \le p < +\infty$. Let $V \subseteq W^{1,p}(I)$ be a SUBSPACE s.t.

(i) V is closed in $W^{1,p}(I)$

(ii) If $u \in V$ is constant, then $u \equiv 0$.

Then there $\exists$ $C>0$

$$\|u\|_{W^{1,p}(I)} \le C \|u'\|_{L^p(I)} , \quad \forall u \in V.$$

In particular $\|u\|_{W^{1,p}(I)}$ and $\|u'\|_{L^p(I)}$ are equivalent norms on V.

(The proof of THEOREM 7.36 can be obtained following the lines of PROOF 2 of THEOREM 7.35. It is left for exercise in the Exercises Course).

EXAMPLE 7.37 We give some examples of subspaces $V \subseteq W^{1,p}$ satisfying the assumptions of THEOREM 7.36:

- $V = \{ u \in W^{1,p}(a,b) \mid u(p)=0 \}$ for $p \in [a,b]$ fixed
  (V is closed by the embedding $W^{1,p}(a,b) \hookrightarrow C[a,b]$)
- $V = \{ u \in W^{1,p}(a,b) \mid \int_a^b u \, dx = 0 \}$
- $V = \{ u \in W^{1,p}(a,b) \mid \int_E u \, dx = 0 \}$, for $E \subseteq [a,b]$ with $|E|>0$

8. EULER-LAGRANGE EQUATION, SOBOLEV CASE

We now analyze variational problems in Sobolev space. First we generalize the following theorems we proved in the C^1 setting: consider the spaces

$$X = \{ u \in C^1[a,b] \mid u(a) = \alpha,\ u(b) = \beta \}, \qquad V = \{ u \in C^1[a,b] \mid u(a) = u(b) = 0 \},$$

the Lagrangian $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x, s, \xi)$, and the functional

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx, \qquad u \in X.$$

① **THEOREM 4.5**: L continuous and C^1 WRT s, ξ.

1) If u_0 minimizes F over X then u_0 solves

(INTEGRAL ELE) $$\boxed{\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v}\, dx = 0, \quad \forall v \in V}$$

2) If $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$ minimizes F over X, then u_0 solves

(ELE) $$\boxed{\begin{cases} \dfrac{d}{dx}\left[L_\xi(x, u_0, \dot{u}_0) \right] = L_s(x, u_0, \dot{u}_0), & \forall x \in [a,b] \\ u_0(a) = \alpha, \quad u_0(b) = \beta \end{cases}}$$

② **THEOREM 5.4**: $L \in C^1$, $u_0 \in X$ solution to (INTEGRAL ELE).

1) If L is CONVEX in s, ξ then u_0 is minimizer of F.

2) If L is STRICTLY CONVEX in s, ξ, then u_0 is the UNIQUE minimizer of F.

We start by relaxing the assumptions on L, by just requiring measurability. Precisely, we will require L to be a Carathéodory function:

DEFINITION 8.1 $\Omega \subseteq \mathbb{R}^d$ open, $L : \Omega \times \mathbb{R}^n \to \mathbb{R}$. We say that L is a CARATHÉODORY FUNCTION if

1) $y \mapsto L(x,y)$ is continuous for a.e. $x \in \Omega$,

2) $x \mapsto L(x,y)$ is Lebesgue measurable for all $y \in \mathbb{R}^n$.

NOTATION Let $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,\xi)$. Whenever we say that the Lagrangian L is Carathéodory we mean that

$$\Omega = (a,b), \quad d = 1, \ n = 2 \quad \text{and} \quad y = (s, \xi)$$

in DEFINITION 8.1.

EXAMPLE $L : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $L(x,s,\xi) := \alpha(x) + g(s,\xi)$ is Carathéodory if $\alpha : (0,1) \to \mathbb{R}$ is measurable and $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

PROPOSITION 8.2 Let $\Omega \subseteq \mathbb{R}^d$ be open, $L : \Omega \times \mathbb{R}^n \to \mathbb{R}$ Carathéodory, $u : \Omega \to \mathbb{R}^n$ measurable. Then $g : \Omega \to \mathbb{R}$ defined by

$$g(x) := L(x, u(x))$$

is measurable.

(Proof is omitted. It is obvious by approximation by step functions – see PROPOSITION 3.7 in the book by Dacorogna).

WEAK EULER-LAGRANGE EQUATION

Let $p \geq 1$, $a < b$, and define the space

$$X := \{ u \in W^{1,p}(a,b) \mid u(a) = \alpha, \ u(b) = \beta \}$$

Note
- X is well-defined, since $W^{1,p}$ functions are continuous by THEOREM 7.19 (so $u(a)$ and $u(b)$ make sense)
- X is an AFFINE space with reference vector space $W_0^{1,p}(a,b)$ (since functions in $W_0^{1,p}(a,b)$ vanish on a, b, by THEOREM 7.34).

Let $L : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x, s, \xi)$ and define $F : W^{1,p}(a,b) \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u, \dot{u}) \, dx$$

WEAK DERIVATIVE

The Sobolev version of THEOREM 4.5 is as follows:

ASSUMPTION 8.3 Assume L, L_s, L_ξ are Carathéodory functions.
Suppose that either of the following holds:

(H1) $\forall R > 0$, $\exists$ $\alpha_1 \in L^1(a,b)$, $\alpha_2 \in L^{p'}(a,b)$, $p' := \frac{p}{p-1}$, $\beta = \beta(R)$ such that $\forall x \in (a,b)$, $|s| \leq R$, $\xi \in \mathbb{R}$

$$|L(x,s,\xi)|, \ |L_s(x,s,\xi)| \leq \alpha_1(x) + \beta |\xi|^p$$

$$|L_\xi(x,s,\xi)| \leq \alpha_2(x) + \beta |\xi|^{p-1}$$

(H2) $\forall R > 0$, $\exists$ $\alpha_1 \in L^1(a,b)$, $\beta = \beta(R)$ such that $\forall x \in (a,b)$, $|s| \leq R$, $\xi \in \mathbb{R}$

$$|L(x,s,\xi)|, \ |L_s(x,s,\xi)|, \ |L_\xi(x,s,\xi)| \leq \alpha_1(x) + \beta |\xi|^p$$

THEOREM 8.4 Suppose the above ASSUMPTION 8.3 holds.
Let $u_0 \in X$ be a minimizer for F over X.

1) If (H1) holds then u_0 satisfies the weak form of ELE

(W-ELE) $$\boxed{\int_a^b L_s(x,u_0,\dot{u}_0)\,v + L_\xi(x,u_0,\dot{u}_0)\,\dot{v}\,dx = 0, \quad \forall\, v \in W_0^{1,p}(a,b)}$$

2) If (H2) holds then u_0 satisfies the weaker form of ELE

(W'-ELE) $$\boxed{\int_a^b L_s(x,u_0,\dot{u}_0)\,v + L_\xi(x,u_0,\dot{u}_0)\,\dot{v}\,dx = 0, \quad \forall\, v \in C_c^\infty(a,b)}$$

3) If in addition $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$ then u_0 satisfies the classical ELE

(ELE) $$\boxed{\frac{d}{dx}\left[L_\xi(x,u_0,\dot{u}_0)\right] = L_s(x,u_0,\dot{u}_0), \quad \forall\, x \in [a,b]}$$

Proof **Step 1. F is well-defined:** let $u \in W^{1,p}(a,b)$. Then both u and $\dot{u}$ are measurable. Since L is Carathéodory, by PROPOSITION 8.2 we get that $g:(a,b) \to \mathbb{R}$ defined by

$$g(x) := L(x, u(x), \dot{u}(x))$$

is measurable. Thus g can be integrated, with the integral possibly being unbounded.

Next we need to show that F is bounded.

Since $W^{1,p}(a,b) \hookrightarrow L^\infty(a,b)$ (THEOREM 7.27), we get $u \in L^\infty(a,b)$. Therefore

$$|u(x)| \le \|u\|_\infty \quad \text{a.e. in } (a,b).$$

Choose $R = \|u\|_\infty$ in (H1) or (H2), so that there exist $\alpha_1 \in L^1(a,b)$, $\beta = \beta(R)$ s.t.

$$(*) \qquad |L(x,s,\xi)| \le \alpha_1(x) + \beta|\xi|^p, \quad \forall x \in (a,b),\ |s| \le \|u\|_\infty,\ \xi \in \mathbb{R}.$$

Thus $(x, u(x), \dot{u}(x)) \in (a,b) \times [-\|u\|_\infty, \|u\|_\infty] \times \mathbb{R}$, and

$$|F(u)| \le \int_a^b |L(x, u(x), \dot{u}(x))|\, dx$$

$$\overset{(*)}{\le} \int_a^b \alpha_1(x)\,dx + \beta \int_a^b |\dot{u}|^p\,dx < +\infty \qquad \alpha_1 \in L^1,\ \dot{u} \in L^p$$

showing that F is well-defined.

Step 2. Gâteaux derivative of F:

CASE OF (H1) : Assume (H1). We show that for every $u \in W^{1,p}$ the functional F is Gâteaux differentiable in every direction $v \in W^{1,p}$, by proving that

$$(**) \qquad \lim_{\varepsilon \to 0} \frac{F(u+\varepsilon v) - F(u)}{\varepsilon} = \int_a^b L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\, \dot{v}\, dx$$

Since we are assuming that L_s, L_ξ are Carathéodory, this means that the maps

$$(s,\xi) \mapsto L_s(x,s,\xi) \quad , \quad (s,\xi) \mapsto L_\xi(x,s,\xi)$$

are continuous for a.e $x \in (a,b)$ fixed. Therefore we can apply the standard chain rule to conclude that the map

$$t \longmapsto L(x, u + t\varepsilon v, \dot{u} + \varepsilon t \dot{v})$$

is differentiable, with

$$(**) \quad \frac{d}{dt}\left\{ L(x, u+t\varepsilon v, \dot{u}+\varepsilon t\dot{v}) \right\} = \varepsilon L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, v + \varepsilon L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, \dot{v}$$

Now set

$$g(x,\varepsilon) := \int_0^1 L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, v + L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, \dot{v} \, dt$$

Then

def of F

$$\frac{1}{\varepsilon}\left\{ F(u+tv) - F(u) \right\} = \frac{1}{\varepsilon} \int_a^b \left\{ L(x, u+tv, \dot{u}+t\dot{v}) - L(x,u,\dot{u}) \right\} dx$$

$$\left(\text{Fundamental Thm of Calculus}\right) \quad = \frac{1}{\varepsilon} \int_a^b \left[\int_0^1 \frac{d}{dt}\left\{ L(x, u+t\varepsilon v, \dot{u}+\varepsilon t\dot{v}) \right\} dt \right] dx$$

$$\left(\text{by } (**) \text{ and def of } g\right) \quad = \int_a^b g(x,\varepsilon)\, dx$$

In order to prove (**) it is then sufficient to show that

(C) $$\lim_{\varepsilon \to 0} \int_a^b g(x,\varepsilon)\,dx = \int_a^b \underbrace{L_s(x,u,\dot{u})\,v + L_\xi(x,u,\dot{u})\,\dot{v}}_{= g(x,0) \text{ by definition of } g}\,dx$$

IDEA To show (C) we use DOMINATED CONVERGENCE: i.e., we need to show

(A) $$\lim_{\varepsilon \to 0} g(x,\varepsilon) = g(x,0) \quad \text{for a.e. } x \in (a,b)$$

(B) $$\sup_{0<\varepsilon<1} |g(x,\varepsilon)| \leq |\Lambda(x)| \quad \text{for a.e. } x \in (a,b), \text{ for some } \Lambda \in L^1(a,b)$$

To do that, first notice that by the embedding $W^{1,p}(a,b) \hookrightarrow L^\infty(a,b)$ we get $u + \varepsilon t v \in L^\infty(a,b)$ for all $\varepsilon > 0$, $t \in [0,1]$.
In particular, for $0 < \varepsilon < 1$, $t \in [0,1]$ we get

(B) $$|u(x) + \varepsilon t v(x)| \leq \|u\|_\infty + \|v\|_\infty \quad \text{a.e. on } (a,b).$$

Thus set $R := \|u\|_\infty + \|v\|_\infty$ in (H1), to obtain the existence of $\alpha_1 \in L^1(a,b)$, $\alpha_2 \in L^{p'}(a,b)$, $\beta = \beta(R)$ s.t.

(1) $$|L(x,s,\xi)| \leq \alpha_1(x) + \beta|\xi|^p$$

(2) $$|L_s(x,s,\xi)| \leq \alpha_1(x) + \beta|\xi|^p \quad , \quad \forall x \in (a,b),\ |s| \leq R,\ \xi \in \mathbb{R}$$

(3) $$|L_\xi(x,s,\xi)| \leq \alpha_2(x) + \beta|\xi|^{p-1}$$

We now show (A) : need DOMINATED CONVERGENCE, as $g(x,\varepsilon)$ is itself an integral.
For a.e. $x \in (a,b)$ we know that the maps

$$(s,\xi) \mapsto L_s(x,s,\xi), \qquad (s,\xi) \mapsto L_\xi(x,s,\xi)$$

are continuous (as L_s, L_ξ Carathéodory). Therefore

$$\lim_{\varepsilon \to 0} \left\{ L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon \dot{v})\, v + L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\,\dot{v} \right\} =$$

$$= L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\,\dot{v}$$

for all $t \in [0,1]$ and a.e. $x \in (0,1)$.
Moreover, as $u+t\varepsilon v$ satisfies (B), we can invoke (2) to get

$$|L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, v| \overset{(2)}{\leq} \left[\alpha_1(x) + \beta\, |\dot{u}+t\varepsilon\dot{v}|^p\right] |v|$$

$$\left(\begin{array}{l}\text{as } \varepsilon, t \in (0,1), \text{ and using} \\ (a+b)^p \leq 2^{p-1}(a^p+b^p) \text{ for } p \geq 1\end{array}\right) \leq \left[\alpha_1(x) + \beta\, 2^{p-1}\left(|\dot{u}|^p + |\dot{v}|^p\right)\right] |v|, \quad \forall\, t \in [0,1]$$

and the RHS belongs to $L^1(0,1)$ since x is fixed.
Similarly, using (3), one also shows that

$$|L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\,\dot{v}| \leq C(x), \qquad \forall\, t \in [0,1]$$

so that $C(x) \in L^1(0,1)$, being a constant (x is fixed). Then by DOMINATED CONVERGENCE

$$\lim_{\varepsilon \to 0} g(x,\varepsilon) = \lim_{\varepsilon \to 0} \int_0^1 L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, v + L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\,\dot{v}\, dt$$

$$= \int_0^1 L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\,\dot{v}\, dt = g(x,0)$$

showing (A).

We now prove Ⓑ: we need to estimate $g(x,\varepsilon)$:

$$|g(x,\varepsilon)| \leq \int_0^1 |L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, v|\, dt + \int_0^1 |L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\,\dot{v}|\, dt$$

For the first integral we use (2):

$$\int_0^1 |L_s(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\, v|\, dt \overset{(2)}{\leq} \int_0^1 \left[\alpha_1(x) + \beta|\dot{u}+t\varepsilon\dot{v}|^p\right]|v(x)|\, dt$$

(as $\varepsilon, t \in (0,1)$ and using $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for $p \geq 1$)

$$\leq \int_0^1 \left[\alpha_1(x) + \beta 2^{p-1}(|\dot{u}|^p + |\dot{v}|^p)\right]|v(x)|\, dt$$

(as nothing depends on t anymore)

$$= \left[\alpha_1(x) + \beta 2^{p-1}(|\dot{u}|^p + |\dot{v}|^p)\right]|v(x)|$$

(since $v \in W^{1,p} \hookrightarrow L^\infty$)

$$\leq \underbrace{\left[\alpha_1(x) + \beta 2^{p-1}(|\dot{u}|^p + |\dot{v}|^p)\right]}_{\in L^1(a,b) \text{ since } \alpha_1 \in L^1(a,b),\ \dot{u}, \dot{v} \in L^p(a,b)}\|v\|_\infty$$

For the second integral we use (3):

$$\int_0^1 |L_\xi(x, u+t\varepsilon v, \dot{u}+t\varepsilon\dot{v})\,\dot{v}|\, dt \overset{(3)}{\leq} \int_0^1 \left[\alpha_2(x) + \beta|\dot{u}+\varepsilon t\dot{v}|^{p-1}\right]|\dot{v}|\, dt$$

(the first term does not depend on t) →

$$= \underbrace{\alpha_2(x)\,|\dot{v}(x)|}_{\in L^1(a,b) \text{ by Hölder as } \alpha_2 \in L^{p'},\ \dot{v} \in L^p} + \underbrace{\beta|\dot{v}(x)|\int_0^1 |\dot{u}+\varepsilon t\dot{v}|^{p-1}\, dt}_{\text{This one is estimated separately below}}$$

$$\beta|\dot{v}(x)|\int_0^1 |\dot{u}+\varepsilon t\dot{v}|^{p-1}\, dt \leq \sup_{t\in[0,1]} \underbrace{\beta|\dot{v}(x)|\,|\dot{u}(x)+\varepsilon t\dot{v}(x)|^{p-1}}_{\substack{\in L^1(a,b) \text{ by Hölder, as} \\ |\dot{u}+\varepsilon t\dot{v}(x)|^{p-1} \in L^{p'}(a,b) \text{ since } \dot{u}+\varepsilon t\dot{v} \in L^p}}$$

Thus, $\exists\, \Lambda \in L^1(a,b)$ s.t. $|g(x,\varepsilon)| \leq \Lambda(x)$ for a.e. $x \in (a,b)$, $\forall\, 0<\varepsilon<1$, showing Ⓑ.

Using the same argument of PROPOSITION 2.3 it is immediate to check that the above implies

$$F_g'(u_0)(v) = 0 .$$

Therefore we conclude that

$$\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \; dx = 0 \quad , \quad \forall v \in W_0^{1,p}(a,b)$$

proving that u_0 solves (W-ELE).

- Assume (H2). For what already proved, F is gâteaux differentiable at u_0 $\forall$ directions in $C^\infty(a,b)$, with

$$F_g'(u_0)(v) = \int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \; dx \quad , \quad \forall v \in C^\infty(a,b)$$

Let $v \in C_c^\infty(a,b)$ be arbitrary. Thus $u_0 + \varepsilon v \in X$, $\forall \varepsilon \in \mathbb{R}$ (as $v(a) = v(b) = 0$)
Since u_0 is a minimizer, as above we can show $F_g'(u_0)(v) = 0$, i.e.

$$\int_a^b L_s(x, u_0, \dot{u}_0)\, v + L_\xi(x, u_0, \dot{u}_0)\, \dot{v} \; dx = 0 \quad , \quad \forall v \in C_c^\infty(a,b)$$

proving that u_0 solves (W'-ELE).

Then (C) follows by DOMINATED CONVERGENCE, showing that F is gâteaux diff. at each $u \in W^{1,p}(a,b)$ with

$$F_g'(u)(v) = \int_a^b L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\, \dot{v} \, dx \quad , \quad \forall v \in W^{1,p}(a,b)$$

CASE OF (H2) : Assume (H2). By similar arguments we can show that F is Gâteaux differentiable for every $u \in W^{1,p}$, in every direction $v \in C^\infty(a,b)$, with

$$F_g'(u)(v) = \int_a^b L_s(x,u,\dot{u})\, v + L_\xi(x,u,\dot{u})\, \dot{v} \, dx$$

The difference WRT the case of (H1) is that now the bound on L_ξ is different, but since $v \in C^\infty(a,b)$ (not $v \in W^{1,p}$ as in the previous case) all the estimates work.

Step 3. Show ELE : Suppose now that $u_0 \in X$ minimizes F over X.

- Assume (H1). For what already proved, F is gâteaux differentiable at u_0 $\forall$ directions in $W^{1,p}(a,b)$, with

$$F_g'(u_0)(v) = \int_a^b L_s(x,u_0,\dot{u}_0)\, v + L_\xi(x,u_0,\dot{u}_0)\, \dot{v} \, dx \quad , \quad \forall v \in W^{1,p}(a,b)$$

Let $v \in W_0^{1,p}(a,b)$ be arbitrary. Thus $u_0 + \varepsilon v \in X$, $\forall \varepsilon \in \mathbb{R}$ (as $v(a) = v(b) = 0$)
Since u_0 is a minimizer, we get

$$F(u_0) \leq F(u_0 + \varepsilon v)$$

- Assume that in addition $L \in C^2$ and $u_0 \in X \cap C^2[a,b]$. Since L satisfies at least one between (H1) and (H2) by assumption, we deduce that u_0 solves either (W-ELE) or (W'-ELE). In both cases, we have

$$\int_a^b L_s(x,u_0,\dot{u}_0)\,v + L_\xi(x,u_0,\dot{u}_0)\,\dot{v}\;dx = 0 \qquad , \qquad \forall v \in C_c^\infty(a,b)$$

As u_0 and L are C^2, we can integrate by parts the above, and use that $v(a)=v(b)=0$ to get

$$\int_a^b \left\{ L_s(x,u_0,\dot{u}_0) - \left[L_\xi(x,u_0,\dot{u}_0)\right]' \right\} v\;dx = 0 \qquad , \qquad \forall v \in C_c^\infty(a,b)$$

By the standard FLCV LEMMA 3.4 we deduce (ELE) $\square$