LESSON 9 - 12 MAY 2021

SOBOLEV EMBEDDING

DEFINITION X, Y normed spaces, $X \subseteq Y$. We say that

1. X EMBEDS continuously in Y, in symbols $X \hookrightarrow Y$, if the identity $i:(X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is continuous, i.e. if $\exists\, C>0$ s.t.

$$\|u\|_Y \leq C \|u\|_X, \quad \forall u \in X.$$

2. The embedding $X \hookrightarrow Y$ is COMPACT if the identity $i: X \to Y$ is a continuous compact operator, i.e.,

$$\text{If } B \subseteq X \text{ is norm bounded WRT } \|\cdot\|_X \Rightarrow \overline{B}^{\|\cdot\|_Y} \text{ is compact WRT } \|\cdot\|_Y.$$

THEOREM 7.27 (Sobolev embedding)

Let $I \subseteq \mathbb{R}$ be open. There $\exists\, C>0$, depending only on $|I|$, s.t.

$$\|u\|_{L^\infty(I)} \leq C \|u\|_{W^{1,p}(I)}, \quad \forall u \in W^{1,p}(I), \quad 1 \leq p \leq +\infty.$$

Thus $W^{1,p}(I) \hookrightarrow L^\infty(I)$. If in addition I is BOUNDED:

(a) The embedding $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is COMPACT $\forall\, 1 < p \leq +\infty$,

(b) The embedding $W^{1,1}(I) \hookrightarrow L^q(I)$ is COMPACT $\forall\, 1 \leq q < +\infty$,

(c) The embedding $W^{1,p}(I) \hookrightarrow L^p(I)$ is COMPACT $\forall\, 1 \leq p \leq +\infty$.

In order to prove THEOREM 7.27 we need two auxiliary results:

THEOREM 7.28 (ASCOLI - ARZELÀ)

Let (K, d) be a compact metric space, and consider $C(K)$ i.e. the set of continuous functions $u: K \to \mathbb{R}$. Let $A \subseteq C(K)$ and suppose that:

1. A is BOUNDED: i.e. $M > 0$ s.t. $\|u\|_\infty \leq M$ for all $u \in A$
2. A is EQUI-CONTINUOUS: i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$d(x_1, x_2) < \delta \Rightarrow |u(x_1) - u(x_2)| < \varepsilon, \quad \forall u \in A.$$

Then the closure of A in $C(K)$ is COMPACT.

(This theorem should already be well-known in euclidean spaces. For a proof of the metric case, see the book by RUDIN.)

For the next result, recall the notation: if $u: \mathbb{R} \to \mathbb{R}$, $h \in \mathbb{R}$, the TRANSLATION operator T_h is defined by $(T_h u)(x) := u(x+h)$.

THEOREM 7.29 (Characterization of Sobolev Functions)

Let $1 < p < +\infty$ and $u \in L^p(\mathbb{R})$. They are equivalent:

(a) $u \in W^{1,p}(\mathbb{R})$

(b) It holds $\|T_h u - u\|_{L^p(\mathbb{R})} \leq \|u'\|_{L^p(\mathbb{R})} |h|$, $\forall h \in \mathbb{R}$.

Moreover, the implication (a) $\Rightarrow$ (b) is also true for $p = 1$.

(The proof of the above theorem will be left for the exercises course)

Proof of THEOREM 7.27 We start by showing the embedding $W^{1,p}(I) \hookrightarrow L^\infty(I)$. WLOG we can suppose $I = \mathbb{R}$, otherwise we can use the extension operator of THEOREM 7.24. Also, the embedding is trivial for $p = +\infty$. Hence assume $1 \leq p < +\infty$. Define $G(s) := |s|^{p-1} s$. Let $u \in C^1_c(\mathbb{R})$ and set

$$w := G(u).$$

Clearly $w \in C^1_c(\mathbb{R})$, with (w is compactly supported since $u \in C^1_c(\mathbb{R})$ and $G(0) = 0$)

$$w' = G'(u) u' = p |u|^{p-1} u'$$

Therefore for $x \in \mathbb{R}$,

$$G(u(x)) = w(x) = \int_{-\infty}^{x} w'(s)\, ds$$ (by the Fundamental Theorem of Calculus, since $w \in C^1_c(\mathbb{R})$)

$$(*) \quad = \int_{-\infty}^{x} p |u(s)|^{p-1} u'(s)\, ds$$

Now $|G(u)| = |u|^p$, thus, by $(*)$ and Hölder's inequality,

$$|u(x)|^p = |G(u(x))| \overset{(*)}{\leq} \int_{-\infty}^{x} p |u(s)|^{p-1} |u'(s)|\, ds$$

(Since integrand is non-negative) → $$\leq p \int_{\mathbb{R}} |u(s)|^{p-1} |u'(s)|\, ds$$

(Hölder WRT p, p') $$\leq p \left(\int_{\mathbb{R}} |u|^{p'(p-1)}\, ds \right)^{1/p'} \left(\int_{\mathbb{R}} |u'|^p\, ds \right)^{1/p}$$

(Recall $p' = \frac{p}{p-1}$. Then $p'(p-1) = p$ and $\frac{1}{p'} = \frac{p-1}{p}$) $$= p \|u\|_{L^p}^{p-1} \|u'\|_{L^p}$$

Therefore

$$|u(x)| \leq p^{1/p} \|u\|_{L^p}^{1/p'} \|u'\|_{L^p}^{1/p}, \quad \forall x \in \mathbb{R}.$$

Recall Young's Inequality for real numbers: $ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$, $\forall a,b \ge 0$. Apply it to $a = \|u'\|_{L^p}^{1/p}$, $b = \|u\|_{L^p}^{1/p'}$ to get

$$|u(x)| \le p^{1/p} \|u\|_{L^p}^{1/p'} \|u'\|_{L^p}^{1/p}$$

$$\text{(Young)} \quad \le p^{1/p} \left\{ \frac{\|u\|_{L^p}}{p'} + \frac{\|u'\|_{L^p}}{p} \right\}$$

$$\text{(Since } p, p' \ge 1) \quad \le p^{1/p} \left\{ \|u\|_{L^p} + \|u'\|_{L^p} \right\} = p^{1/p} \|u\|_{W^{1,p}}$$

Taking the supremum for $x \in \mathbb{R}$ and noting that $p^{1/p} \le e^{1/e}$ $\forall p \ge 1$, we get

$$(**) \qquad \|u\|_{L^\infty} \le C \|u\|_{W^{1,p}}, \quad \forall u \in C_c^1(\mathbb{R})$$

with $C := e^{1/e}$. Suppose now that $u \in W^{1,p}(\mathbb{R})$. By THEOREM 7.24 there $\exists \{u_n\} \subseteq C_c^1(\mathbb{R})$ s.t. $u_n \to u$ strongly in $W^{1,p}(\mathbb{R})$. By applying $(**)$ to $(u_n - u_m) \in C_c^1(\mathbb{R})$ we have

$$\|u_n - u_m\|_{L^\infty} \le C \|u_n - u_m\|_{W^{1,p}} \to 0 \quad \text{as } n, m \to +\infty,$$

Since u_n is convergent in $W^{1,p}(\mathbb{R})$ and so it is Cauchy, being $W^{1,p}$ complete (see PROPOSITION 7.16). Therefore $\{u_n\}$ is a Cauchy sequence in $L^\infty(\mathbb{R})$.
As $L^\infty(\mathbb{R})$ is complete, we conclude the $\exists$ of $\tilde{u} \in L^\infty(\mathbb{R})$ s.t. $u_n \to \tilde{u}$ strongly in $L^\infty(\mathbb{R})$.
Recalling that $u_n \to u$ in $W^{1,p}(\mathbb{R})$, we immediately conclude that $\tilde{u} = u$.
By $(**)$ we have

$$\|u_n\|_{L^\infty} \le C \|u_n\|_{W^{1,p}}, \quad \forall n \in \mathbb{N}$$

Since $u_n \to u$ in $L^\infty(\mathbb{R})$ and in $W^{1,p}(\mathbb{R})$, we can pass to the limit as $n \to +\infty$ in the above and obtain our thesis:

$$\|u\|_{L^\infty} \le C \|u\|_{W^{1,p}}, \quad \forall u \in W^{1,p}(\mathbb{R}).$$

(a) Assume I bounded. We need to prove that the embedding

$$W^{1,p}(I) \hookrightarrow C(\bar{I})$$

is compact, for all $1 < p \leq +\infty$. Therefore let $B \subseteq W^{1,p}(I)$ be a bounded set, so that there $\exists\, M > 0$ s.t.

$$\|u\|_{W^{1,p}} \leq M, \qquad \forall u \in B.$$

By the embedding we just proved, it follows that

$$\|u\|_{L^\infty} \leq C \|u\|_{W^{1,p}} \leq CM, \qquad \forall u \in B.$$

Recalling that $W^{1,p}(I) \subseteq C(\bar{I})$ (see THEOREM 7.19), we get $\|u\|_\infty = \|u\|_{L^\infty}$, so that

$$(*) \qquad \|u\|_\infty \leq CM, \qquad \forall u \in B.$$

Moreover, by THEOREM 7.23 we have $W^{1,p}(I) \subseteq C^{0,1-1/p}(I)$, if $p > 1$, with

$$|u(x) - u(y)| \leq \|\dot{u}\|_{L^p} |x-y|^{1-1/p}, \qquad \forall x, y \in \bar{I}.$$

As $\|\dot{u}\|_{L^p} \leq M$ for all $u \in B$, we conclude that

$$(**) \qquad |u(x) - u(y)| \leq M |x-y|^{1-1/p}, \qquad \forall x, y \in \bar{I}, \quad \forall u \in B$$

which shows that the family $B \subseteq C(\bar{I})$ is EQUI-CONTINUOUS. As $(*)$ - $(**)$ hold, we can apply the ASCOLI - ARZELÀ THEOREM 7.28 with $K = \bar{I}$, to conclude that $\bar{B}$ is compact in $C(\bar{I})$ (where the closure is taken WRT the uniform norm in $C(\bar{I})$). Thus, (a) is established.

(b) Let I be bounded, $1 \leq q < +\infty$. We need to prove that the embedding

$$W^{1,1}(I) \hookrightarrow L^q(I)$$

is compact. So let $B \subseteq W^{1,1}(I)$ be a bounded set, i.e.

$$\|u\|_{W^{1,1}(I)} \leq M, \quad \forall u \in B.$$

Let $P: W^{1,1}(I) \to W^{1,1}(\mathbb{R})$ be the extension operator from LEMMA 7.25.
By the properties of P, the set $P(B)$ is bounded in $W^{1,1}(\mathbb{R})$, and also $P(B)|_I = B$, where

$$P(B)|_I := \{ u : I \to \mathbb{R} \mid \exists v \in P(B) \text{ s.t. } v|_I = u \}.$$

By the embedding $W^{1,1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ we already proved, we have that $P(B)$ is also bounded in $L^\infty(\mathbb{R})$. Then, for $u \in P(B)$ we have

$$\int_{\mathbb{R}} |u|^q \, dx = \int_{\mathbb{R}} |u|^{q-1} |u| \, dx \leq \|u\|_{L^\infty}^{q-1} \|u\|_{L^1}, \qquad \forall u \in P(B)$$

showing that $P(B)$ is also bounded in $L^q(\mathbb{R})$, i.e., $\exists M > 0$ s.t.

$$(*) \qquad \|u\|_{L^q(\mathbb{R})} \leq M, \qquad \forall u \in P(B).$$

We now check that

$$(**) \qquad \lim_{|h| \to 0} \sup_{u \in P(B)} \|\tau_h u - u\|_{L^q(\mathbb{R})} = 0.$$

Indeed, by THEOREM 7.29 (implication (a) $\Rightarrow$ (b) with $p=1$) we have

$$(***) \qquad \|\tau_h u - u\|_{L^1(\mathbb{R})} \leq \|\dot{u}\|_{L^1(\mathbb{R})} |h| \leq C|h|, \qquad \forall u \in P(B)$$

Since $P(B)$ is bounded in $W^{1,1}(\mathbb{R})$

Therefore, for $u \in P(B)$,

$$\|\tau_h u - u\|^q_{L^q(\mathbb{R})} = \int_{\mathbb{R}} |\tau_h u - u|^q \, dx = \int_{\mathbb{R}} |\tau_h u - u|^{q-1} \, |\tau_h u - u| \, dx$$

$$\leq \|\tau_h u - u\|^{q-1}_{L^\infty(\mathbb{R})} \, \|\tau_h u - u\|_{L^1(\mathbb{R})}$$

$$\left(\|\tau_h u - u\|_{L^\infty} \leq 2\|u\|_{L^\infty}\right) \quad \leq 2^{q-1} \|u\|^{q-1}_{L^\infty(\mathbb{R})} \, \|\tau_h u - u\|_{L^1(\mathbb{R})}$$

$$\left(\begin{array}{l}\text{By } (***) \text{ and the fact} \\ \text{that } P(B) \text{ is bounded in } L^\infty(\mathbb{R})\end{array}\right) \quad \leq 2^{q-1} \tilde{C} \, C \, |h|$$

showing (**). Since (*) - (**) hold, and I bounded, $q \neq +\infty$, we can apply FRÉCHET-KOLMOGOROV THEOREM 6.17 to conclude that the closure of $P(B)|_I$ is compact in $L^q(I)$. Recalling that $P(B)|_I = B$, we have that the closure of B is compact in $L^q(I)$.

(c) Let $I \subseteq \mathbb{R}$ be bounded. We are left to show that the embedding

$$(*) \qquad W^{1,p}(I) \hookrightarrow L^p(I)$$

is compact for every $1 \leq p \leq +\infty$. Indeed, for $p = 1$, (*) is just a special case of (b) with $q = 1$. Instead, for $1 < p \leq +\infty$, (*) follows from the compact embedding $W^{1,p}(I) \hookrightarrow C(\bar{I})$ of point (a), and from the fact that uniform convergence implies L^p convergence. □

REMARK 7.30

We want to discuss (without proof) what happens in the cases left out from THEOREM 7.27.

(1) For the compact embedding $W^{1,p}(I) \hookrightarrow C(\bar{I})$, I bounded, $1 < p \leq +\infty$:

- Let I be bounded. We have that $W^{1,1}(I)$ embeds into $C(\bar{I})$ (by THEOREM 7.9), but the embedding is in general NOT compact.

- What kind of compactness can we expect in this case? The answer is as follows: Let $I \subseteq \mathbb{R}$ be open (bounded or unbounded). If $\{u_n\} \subseteq W^{1,1}(I)$ is bounded, there exists a subsequence u_{n_k} s.t. $u_{n_k}(x)$ converges pointwise for all $x \in \bar{I}$

  (this is called HELLY'S SELECTION THEOREM)

(2) Concerning the embedding $W^{1,p}(I) \hookrightarrow L^\infty(I)$ for all $1 \leq p \leq +\infty$:

- When I is unbounded, the above embedding is NEVER COMPACT

- Assume I unbounded and $1 < p \leq +\infty$. If $\{u_n\} \subseteq W^{1,p}(I)$ is bounded, then $\exists\, u \in W^{1,p}(I)$ and a subsequence s.t. $u_{n_k} \to u$ in $L^\infty(J)$ for every $J \subseteq I$ bounded.

(3) Let I be unbounded. Then $W^{1,p}(I) \hookrightarrow L^q(I)$ for all $q \in [p, \infty]$. However, in general, $W^{1,p}(I)$ does NOT embed into $L^q(I)$ if $q \in [1, p)$.

We want to explicitely state a Corollary of THEOREM 7.27 regarding weak convergence. To this end, we first recall the general definition of compact operator.

DEFINITION Let X, Y be normed spaces, and $T \in \mathcal{J}(X, Y)$. We say that T is **COMPACT** if it holds:

$$B \subseteq X \text{ bounded WRT } \|\cdot\|_X \Rightarrow \overline{T(B)}^{\|\cdot\|_Y} \text{ compact WRT } \|\cdot\|_Y$$

PROPOSITION 7.31 Let X, Y be normed spaces, and $T \in \mathcal{J}(X, Y)$ be compact. It holds:

$$x_n \rightharpoonup x_0 \text{ weakly in } X \Rightarrow Tx_n \to Tx_0 \text{ strongly in } Y$$

Proof: Assume $x_n \rightharpoonup x_0$ weakly in X. Since X is a normed space, we have that $\{x_n\}$ is bounded WRT $\|\cdot\|$.

Thus, by definition of compact operator, $\overline{\{Tx_n\}}^{\|\cdot\|_Y}$ is compact WRT $\|\cdot\|_Y$. Therefore, as $\{Tx_n\} \subseteq \overline{\{Tx_n\}}^{\|\cdot\|_Y}$, there $\exists$ a subsequence and $y \in Y$ s.t.

$$(*) \quad Tx_{n_k} \to y \quad \text{strongly in } Y.$$

Now, we know that $x_n \rightharpoonup x_0$ and T continuous. Thus (easy check)

$$(**) \quad Tx_n \rightharpoonup Tx_0 \quad \text{weakly in } Y.$$

Since $(*)$ holds, and strong convergence implies weak convergence, we get $Tx_{n_k} \rightharpoonup y$ weakly in Y. By $(**)$ and uniqueness of the weak limit we get $y = Tx_0$. Therefore $(*)$ reads

$$(***) \quad Tx_{n_k} \to Tx_0 \quad \text{strongly in } Y.$$

To conclude, we use the following standard fact:

FACT (X,τ) topological space, $\{x_n\} \subseteq X$. Suppose that for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a subsequence $\{x_{n_{k_j}}\}$ such that

$$x_{n_{k_j}} \to x_0 \quad \text{as } j \to +\infty,$$

for some $x_0 \in X$ which does not depend on the subsequence $\{x_{n_k}\}$ chosen. Then $x_n \to x_0$.

Therefore, reasoning as above, we could have started from an arbitrary subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$, and shown that $\exists$ $\{Tx_{n_{k_j}}\}$ such that

$$Tx_{n_{k_j}} \to Tx_0 \text{ strongly in } Y, \text{ as } j \to +\infty.$$

Since the limit does not depend on the chosen subsequence $\{Tx_{n_k}\}$, we conclude that $Tx_n \to Tx_0$ strongly in Y. $\square$

COROLLARY 7.32 Let $I=(a,b)$ be bounded, and $1 \le p < +\infty$. If

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(a,b)$$

(i.e., $u_n \rightharpoonup u$, $\dot{u}_n \rightharpoonup \dot{u}$ weakly in $L^p(a,b)$), then

$$u_n \to u \text{ strongly in } L^p(a,b).$$

Proof By point (c) of THEOREM 7.27 we have that $W^{1,p}(a,b) \hookrightarrow L^p(a,b)$ is compact for every $1 \le p \le +\infty$. The thesis follows by applying PROPOSITION 7.31. $\square$