Then we can pass to the limit in (*) and get that $\dot{u} = g$ in the weak sense.
As $g \in L^2(I)$, we get $u \in H^1(I)$. If $v \in H^1(I)$ we get

$$\langle u_n, v \rangle_{H^1} = \langle u_n, v \rangle_{L^2} + \langle \dot{u}_n, \dot{v} \rangle_{L^2} \to \langle u, v \rangle_{L^2} + \langle \dot{u}, \dot{v} \rangle_{L^2} = \langle u, v \rangle_{H^1}$$

showing that $u_n \rightharpoonup u$ weakly in $H^1(I)$. $\square$

LESSON 8 - 5 MAY 2021

We now prove one of the main results on 1-dimensional Sobolev functions, namely, that they are CONTINUOUS and they are PRIMITIVES of L^p functions.

THEOREM 7.19 Let $I = (a,b)$ be bounded or unbounded, and $1 \le p \le +\infty$.
Let $u \in W^{1,p}(I)$. Then there $\exists\, \tilde{u} \in C(I)$ s.t.

$$u = \tilde{u} \quad \text{a.e. on } I$$

and

$$(*) \qquad \tilde{u}(x) - \tilde{u}(y) = \int_y^x \dot{u}(t)\,dt \,, \qquad \forall\, x,y \in I.$$

(Generalized Fundamental Thm of Calculus)

NOTE Theorem 7.19 is saying that if $u \in W^{1,p}(I)$ then $\exists\, \tilde{u}$ continuous in the same equivalence class of u. We call $\tilde{u}$ the CONTINUOUS REPRESENTATIVE of u, and in the future we just denote it by u (Notice that the continuous representative is unique, by (*)).

During the proof of THEOREM 7.19 we need the following lemma.

LEMMA 7.20 $I = (a,b)$, $g \in L^1_{loc}(I)$. Fix $y_0 \in I$ and define

$$u(x) := \int_{y_0}^x g(t)\,dt \,, \qquad \forall\, x \in I.$$

Then $u \in C(I)$ and $\dot{u} = g$ in the weak sense.

Proof of LEMMA 7.20 The fact that u is continuous follows by DOMINATED CONVERGENCE. Indeed, for $x \in I$,

$$(*) \quad |u(x+\varepsilon) - u(x)| \leq \int_x^{x+\varepsilon} |g(t)|\,dt = \int_K \chi_{[x,x+\varepsilon]}(t)\,|g(t)|\,dt,$$

where K is any compact set such that $[x, x+\varepsilon] \subset K$, $\forall\, 0<\varepsilon<1$. Now $\chi_{[x,x+\varepsilon]}\, g \to 0$ a.e. as $\varepsilon \to 0$, and $|\chi_{[x,x+\varepsilon]}\, g| \leq |g|$,

with $g \in L^1_{loc}(I)$. Thus $g \in L^1(K)$, and by dominated convergence we conclude that the RHS of $(*)$ goes to 0 as $\varepsilon \to 0$, showing continuity.

We now show that $\dot{u} = g$ in the weak sense. Thus let $\varphi \in C^1_c(I)$. Consider $\psi(t,x) := u(x)\dot{\varphi}(t)$. Clearly $\psi \in L^1(I \times I)$, being $u, \dot{\varphi}$ continuous. Then we can apply FUBINI'S THEOREM 6.10 to get:

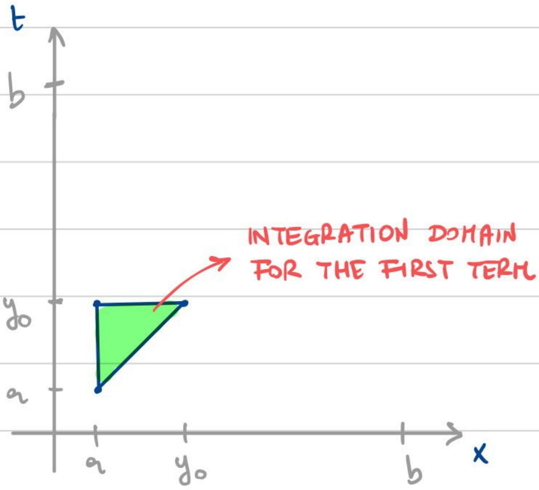
$$\int_a^b u \dot{\varphi}\, dx = \int_a^b \left\{ \int_{y_0}^x g(t)\,\dot{\varphi}(x)\,dt \right\} dx \qquad \text{(definition of } u\text{)}$$

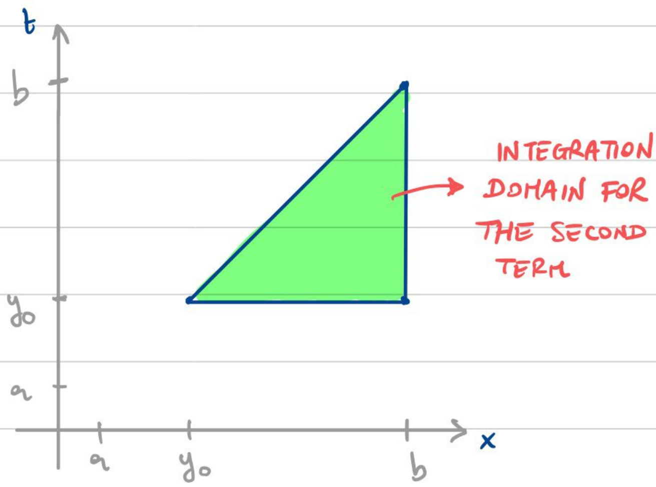
Splitting integral WRT x →

$$= \int_a^{y_0} \left\{ \int_{y_0}^x g(t)\,\dot{\varphi}(x)\,dt \right\} dx + \int_{y_0}^b \left\{ \int_{y_0}^x g(t)\,\dot{\varphi}(x)\,dt \right\} dx$$

Reversing integration values of inner integral for the FIRST TERM

$$= -\int_a^{y_0} \left\{ \int_x^{y_0} g(t)\,\dot{\varphi}(x)\,dt \right\} dx + \int_{y_0}^b \left\{ \int_{y_0}^x g(t)\,\dot{\varphi}(x)\,dt \right\} dx$$





We can write the integration domains normal WRT to t, and apply FUBINI:

$$\int_a^b u \dot{\varphi}\, dx = -\int_a^{y_0} \left\{ \int_x^{y_0} g(t)\, \dot{\varphi}(x)\, dt \right\} dx + \int_{y_0}^b \left\{ \int_{y_0}^x g(t)\, \dot{\varphi}(x)\, dt \right\} dx$$

FUBINI $\rightarrow$
$$= -\int_a^{y_0} \left\{ \int_a^{t} g(t)\, \dot{\varphi}(x)\, dx \right\} dt + \int_{y_0}^b \left\{ \int_{t}^b g(t)\, \dot{\varphi}(x)\, dx \right\} dt$$

TAKE $g(t)$ OUT $\rightarrow$
$$= -\int_a^{y_0} g(t) \left\{ \int_a^{t} \dot{\varphi}(x)\, dx \right\} dt + \int_{y_0}^b g(t) \left\{ \int_{t}^b \dot{\varphi}(x)\, dx \right\} dt$$

$$= -\int_a^{y_0} g(t)\, [\varphi(t) - \varphi(a)]\, dt + \int_{y_0}^b g(t)\, [\varphi(b) - \varphi(t)]\, dt$$

$\varphi(a) = \varphi(b) = 0$, since φ is COMPACTLY SUPPORTED $\rightarrow$
$$= -\int_a^{y_0} g(t)\varphi(t)\, dt - \int_{y_0}^b g(t)\varphi(t)\, dt = -\int_a^b g(t)\varphi(t)\, dt \,,$$

showing that $\dot{u} = g$ in the weak sense and concluding. $\square$

Proof of THEOREM 7.19 Fix $y_0 \in I$ arbitrary and define

$$\hat{u}(x) := \int_{y_0}^{x} \dot{u}(t)\, dt \,, \qquad \forall x \in I .$$

Since $\dot{u} \in L^p(I)$ (as $u \in W^{1,p}(I)$), then $\dot{u} \in L^1_{loc}(I)$. We can then apply LEMMA 7.20 to infer that $\hat{u} \in C(I)$ and $(\hat{u})' = \dot{u}$ in the weak sense, i.e.

$$(*) \quad \int_a^b \hat{u}\, \dot{\varphi}\, dx = -\int_a^b \dot{u}\, \varphi\, dx \,, \qquad \forall \varphi \in C^1_c(I)$$

On the other hand $u \in W^{1,p}(I)$, so that by definition

$$\int_a^b u \dot{\varphi}\, dx = -\int_a^b \dot{u} \varphi\, dx, \quad \forall \varphi \in C_c^1(I).$$

by (*) we then get

$$\int_a^b (\hat{u} - u)\dot{\varphi}\, dx = 0, \quad \forall \varphi \in C_c^1(I).$$

We can then apply DBR LEMMA 7.13 to get that $\exists\, c \in \mathbb{R}$ s.t. $u = \hat{u} + c$ a.e. in I. Thus the continuous representative is $\tilde{u} := \hat{u} + c$. The second part of the statement follows by definition of $\tilde{u}$. □

REMARK 7.21 Lemma 7.20 implies that, if $g \in L^p(I)$ and its primitive u also belongs to $L^p(I)$, then $u \in W^{1,p}(I)$.

With similar ideas, we can prove the following proposition.

PROPOSITION 7.22 Let $I = (a,b)$ be bounded or unbounded, $1 \le p \le +\infty$. Assume that $u \in W^{1,p}(I)$ is s.t. $\dot{u} \in C(I)$. Then $u \in C^1(I)$.

Proof Define $U(x) := \int_a^x \dot{u}(t)\,dt$. As $\dot{u}$ is continuous, by the Fundamental Theorem of Calculus we have that $U \in C^1(I)$ and $\dot{U} = \dot{u}$. Let $\varphi \in C_c^1(I)$. Integrating by parts:

$$\int_a^b U\dot{\varphi}\, dx = \underbrace{U\varphi\Big|_a^b}_{\substack{=0 \text{ as} \\ \varphi(a) = \varphi(b) = 0}} - \int_a^b \dot{U}\varphi\, dx \underset{\dot{U} = \dot{u}}{=} -\int_a^b \dot{u}\varphi\, dx \underset{\text{Definition of weak derivative}}{=} \int_a^b u\dot{\varphi}\, dx$$

Thus

$$\int_a^b (U - u)\dot{\varphi}\, dx = 0, \quad \forall \varphi \in C_c^1(I).$$

By DBR LEMMA 7.13 we get $u = U + c$ for some $c \in \mathbb{R}$. As $U \in C^1(I) \Rightarrow u \in C^1(I)$. □

HÖLDER REGULARITY

We can actually improve on THEOREM 7.19 by showing Hölder regularity for Sobolev functions. We recall that u is **α- HÖLDER** for some $0<\alpha<1$ if $\exists\, C>0$ s.t.

$$|u(x)-u(y)| \leq C\, |x-y|^{\alpha} \quad , \quad \forall\, x,y \in I$$

We denote the space of α- Hölder functions by $C^{0,\alpha}(I)$.

THEOREM 7.23 Let $I=(a,b)$ be bounded or unbounded. Let $1<p\leq +\infty$ and $u \in W^{1,p}(I)$. Then $u \in C^{0,1-1/p}(I)$, with

$$|u(x)-u(y)| \leq \|\dot{u}\|_{L^p}\, |x-y|^{1-1/p} \quad , \qquad \forall\, x,y \in I.$$

Proof By THEOREM 7.19 we have that u is continuous and

$$u(x)-u(y) = \int_y^x \dot{u}(t)\, dt \quad , \quad \forall\, x,y \in I.$$

Then for $y>x$,

$$|u(x)-u(y)| \leq \int_x^y |\dot{u}(t)|\, dt$$

(Hölder inequality) $$\leq \left(\int_x^y |\dot{u}(t)|^p\, dt\right)^{1/p} \left(\int_x^y 1^{p'}\, dt\right)^{1/p'}$$ $\left(p' = \frac{p}{p-1} \text{ Hölder conjugate}\right)$

$$= \left(\int_a^b |\dot{u}(t)|^p\, dt\right)^{1/p} (y-x)^{1/p'}$$

$$= \|\dot{u}\|_{L^p}\, |y-x|^{1-1/p}$$

If $x>y$ we conclude with the same argument. $\square$

WARNING THEOREM 7.23 does not hold for $p=1$.

DENSITY OF SMOOTH FUNCTIONS

Our goal is to prove the following theorem.

THEOREM 7.24 Let $1 \le p < +\infty$, $u \in W^{1,p}(I)$ for $I = (a,b)$ bounded or unbounded. Then $\exists \{u_n\} \subseteq C_c^\infty(\mathbb{R})$ s.t.

$$u_n|_I \to u \quad \text{strongly in } W^{1,p}(I).$$

WARNING The above differs from the density result for L^p functions COROLLARY 7.10: If $u \in L^p(I)$, $\exists \{u_n\} \subseteq C_c^\infty(I)$ s.t. $u_n \to u$ strongly in $L^p(I)$.

In order to prove the above theorem we need an extension result.

LEMMA 7.25 Let $I = (a,b)$ be bounded or unbounded, $1 \le p \le +\infty$. There $\exists$ a linear continuous operator $P: W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ called EXTENSION OPERATOR such that:

a) $Pu|_I = u$, $\quad \forall u \in W^{1,p}(I)$

b) $\|Pu\|_{L^p(\mathbb{R})} \le C \|u\|_{L^p(I)}$, $\quad \forall u \in W^{1,p}(I)$

c) $\|Pu\|_{W^{1,p}(\mathbb{R})} \le C \|u\|_{W^{1,p}(I)}$, $\quad \forall u \in W^{1,p}(I)$

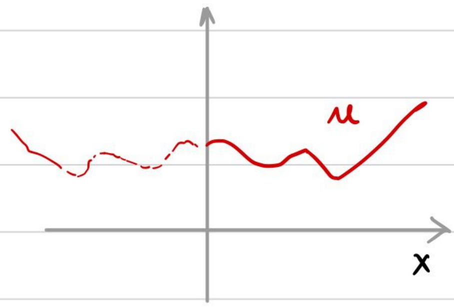
where C depends only on $|I|$: $C = 4$ in (b) and $C = 4\left(1 + \frac{1}{|I|}\right)$ in (c).

Proof of LEMMA 7.25 We have two cases:

1) If I is unbounded, then by translation it is sufficient to consider either $I = (0, +\infty)$, or $I = (-\infty, 0)$.
2) If I is bounded, then by translation and scaling it is sufficient to consider $I = (0,1)$.

CASE 1 Let $I=(0,+\infty)$. If $u\in W^{1,p}(I)$, we extend u by REFLECTION:

$$(Pu)(x) := u^*(x) := \begin{cases} u(x) & \text{if } x>0 \\ u(-x) & \text{if } x<0 \end{cases}$$



Clearly $u^*|_I = u$, so that (a) holds. Also

$$\|u^*\|^p_{L^p(\mathbb{R})} = \int_{\mathbb{R}} |u^*|^p\,dx = 2\int_0^{+\infty} |u|^p\,dx$$

then

$$(*) \quad \|u^*\|_{L^p(\mathbb{R})} \le 2^{1/p}\,\|u\|_{L^p(I)} \le 2\,\|u\|_{L^p(I)},$$

showing (b). Now define

$$g(x) := \begin{cases} \dot u(x) & \text{for a.e. } x>0 \\ -\dot u(-x) & \text{for a.e. } x<0 \end{cases}$$

(use the definition of u^*, doing separately the cases $x>0$ and $x<0$)

Clearly $g\in L^p(\mathbb{R})$. Also, by using THEOREM 7.19, it is easy to check that

$$u^*(x) - u^*(0) = \int_0^x g(t)\,dt, \qquad \forall\, x\in\mathbb{R}.$$

Hence $u^*\in W^{1,p}(\mathbb{R})$ by REMARK 7.21, with $(u^*)' = g$ in the weak sense. Finally

$$\|(u^*)'\|^p_{L^p(\mathbb{R})} = \int_{\mathbb{R}} |g(t)|^p\,dt = 2\int_0^{+\infty} |\dot u(t)|^p\,dt = 2\,\|\dot u\|^p_{L^p(I)},$$

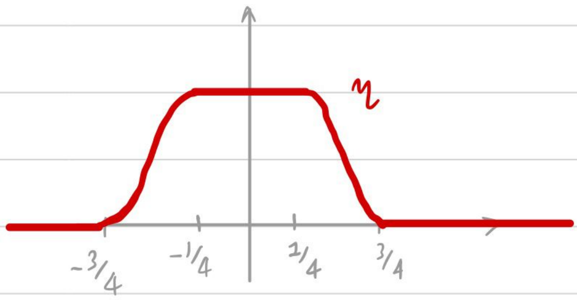
so that

$$\|(u^*)'\|_{L^p(\mathbb{R})} \le 2^{1/p}\,\|\dot u\|_{L^p(I)} \le 2\,\|\dot u\|_{L^p(I)}.$$

Together with $(*)$, this implies (c). The case $I=(-\infty,0)$ is the same.

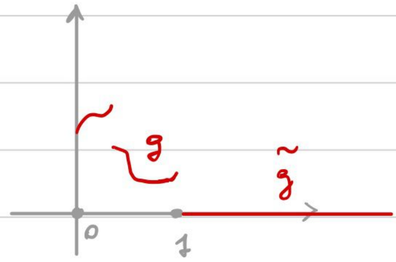
CASE 2 Let $I = (0,1)$. Let $\eta \in C^1_c(\mathbb{R})$ be a cut-off such that

- $0 \leq \eta \leq 1$ in $\mathbb{R}$
- $\eta(x) = 1$ for all $x \in [-1/4, 1/4]$
- $\eta(x) = 0$ for all $x \in \mathbb{R} \setminus (-3/4, 3/4)$.



For $g \in L^p(0,1)$, define

$$\tilde{g}(x) := \begin{cases} g(x) & \text{if } x \in (0,1) \\ 0 & \text{if } x \geq 1 \end{cases}$$



If $u \in W^{1,p}(0,1)$, we claim that

$$\circledast \qquad \eta\tilde{u} \in W^{1,p}(0,+\infty) \quad \text{and} \quad (\eta\tilde{u})' = \eta'\tilde{u} + \eta\cdot\widetilde{(u')} \quad \text{weakly.}$$

Indeed, let $\varphi \in C^1_c(0,+\infty)$. As both φ and η are regular, we have $(\eta\varphi)' = \eta'\varphi + \eta\varphi'$.

Then

$$\int_0^{+\infty} (\eta\tilde{u})\varphi'\,dx = \int_0^1 (\eta u)\varphi'\,dx \qquad \left(\begin{array}{l}\text{since } \tilde{u}=0 \text{ if } x\geq 1 \text{ and } \tilde{u}=u \\ \text{for } 0<x<1\end{array}\right)$$

$$= \int_0^1 u(\eta\varphi)'\,dx - \int_0^1 u\,\eta'\varphi\,dx \qquad (\text{using } (\eta\varphi)' = \eta'\varphi + \eta\varphi')$$

$$= -\int_0^1 u'\,\eta\varphi\,dx - \int_0^1 u\,\eta'\varphi\,dx \qquad \left(\begin{array}{l}\text{since } \eta\varphi \in C^1_c(0,1) \\ \text{and } u \in W^{1,p}(0,1)\end{array}\right)$$

$$= -\int_0^1 [u'\eta + u\eta']\varphi\,dx$$

$$= -\int_0^{+\infty} [\widetilde{(u')}\eta + \tilde{u}\eta']\varphi\,dx \qquad \left(\begin{array}{l}\text{since extending } u \text{ and} \\ u' \text{ to zero does not} \\ \text{alter the integral}\end{array}\right)$$

Showing that $(\eta\tilde{u})' = \eta'\tilde{u} + \eta\,\widetilde{(u')}$ weakly.

Clearly $\eta\tilde{u} \in L^p(0,+\infty)$. Also, by using the formula just proven, $(\eta\tilde{u})' \in L^p(0,+\infty)$. Then $\eta\tilde{u} \in W^{1,p}(0,+\infty)$ and $\circledast$ is proven.

$$(u)^*(x) = \begin{cases} u(x) & \text{if } x>0 \\ u(-x) & \text{if } x<0 \end{cases}$$

We can now define the extension operator $P: W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$. First define $P_1: W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$ by setting $P_1 u := (\eta \tilde{u})^*$, with $*$ being the operator from CASE 1, that is, we first extend u to $(0,+\infty)$ by setting u to 0 in $[1,+\infty)$, then we multiply by η and extend $\eta\tilde{u}$ to $(-\infty,0)$ by reflection. By the properties of $*$ we know that

$$\|(\eta\tilde{u})^*\|_{L^p(\mathbb{R})} \leq 2\,\|\eta\tilde{u}\|_{L^p(0,+\infty)}, \qquad \left\|\left[(\eta\tilde{u})^*\right]'\right\|_{L^p(\mathbb{R})} \leq 2\,\|(\eta\tilde{u})'\|_{L^p(0,+\infty)}$$

Now

$$\|\eta\tilde{u}\|_{L^p(0,+\infty)} \leq \|\eta\|_{L^\infty(0,+\infty)} \|\tilde{u}\|_{L^p(0,+\infty)} = \|u\|_{L^p(0,1)}$$

Since $0 \leq \eta \leq 1$ and $\tilde{u}=0$ in $(1,+\infty)$. Moreover, by $*$,

$$\|(\eta\tilde{u})'\|_{L^p(0,+\infty)} \overset{*}{\leq} \|\eta'\tilde{u}\|_{L^p(0,+\infty)} + \|\eta(\tilde{u}')\|_{L^p(0,+\infty)}$$

(since $u=0$, $(\tilde{u})'=0$ in $(1,+\infty)$)

$$\leq \|\eta'\|_{L^\infty(I)} \|u\|_{L^p(I)} + \|\eta\|_{L^\infty(0,+\infty)} \|u'\|_{L^p(I)}$$

$$\leq C\,\|u\|_{L^p(I)} + \|u'\|_{L^p(I)}$$

with $C := \|\eta'\|_{L^\infty(I)}$. In total, we have

$$(**) \qquad \|P_1 u\|_{L^p(\mathbb{R})} \leq 2\,\|u\|_{L^p(I)}, \qquad \|P_1 u\|_{W^{1,p}(\mathbb{R})} \leq 2(1+C)\,\|u\|_{W^{1,p}(I)}$$

Also notice that $(P_1 u)|_I = \eta u$. Now define $P_2: W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$ in the following way: $P_2 u$ is defined by

- Extending $(1-\eta)u$ to 0 in $(-\infty,0)$, obtaining a map defined in $(-\infty,1]$;
- Then extend to the whole $\mathbb{R}$ by reflection around 1.

In a similar way one can check that

(***) $\| P_2 u \|_{L^p(\mathbb{R})} \leq 2 \| u \|_{L^p(I)}$, $\| P_2 u \|_{W^{1,p}(\mathbb{R})} \leq 2(1+C) \| u \|_{W^{1,p}(I)}$

and that $(P_2 u)|_I = (1-\eta) u$. Finally we define $P : W^{1,p}(0,1) \to W^{1,p}(\mathbb{R})$ by

$$Pu := P_1 u + P_2 u .$$

By (**)-(***) we have that P satisfies (b), (c). Moreover

$$(Pu)|_I = (P_1 u)|_I + (P_2 u)|_I = \eta u + (1-\eta) u = u ,$$

so that also (a) holds, concluding. □

Another result needed to prove THEOREM 7.24 is the following:

LEMMA 7.26 Let $\rho \in L^1(\mathbb{R})$, $u \in W^{1,p}(\mathbb{R})$ with $1 \leq p \leq +\infty$. Then $\rho * u \in W^{1,p}(\mathbb{R})$ and $(\rho * u)' = \rho * u'$ in the weak sense.

Proof Assume first that ρ is compactly supported, so that $\rho \in L^1_{loc}(\mathbb{R})$. By THEOREM 7.2 we have $\rho * u \in L^p(\mathbb{R})$. Let $\varphi \in C^1_c(\mathbb{R})$. One can check that

$$(*) \quad \int_{\mathbb{R}} (\rho * u) \varphi' \, dx = \int_{\mathbb{R}} u (\check{\rho} * \varphi') \, dx , \quad \check{\rho}(x) := \rho(-x) .$$

Now, as $\check{\rho} \in L^1(\mathbb{R})$ and $\varphi \in C^1_c(\mathbb{R})$, by THEOREM 7.6 we have $\check{\rho} * \varphi \in C^1(\mathbb{R})$ and $\check{\rho} * \varphi' = (\check{\rho} * \varphi)'$. Moreover $\check{\rho} * \varphi$ is compactly supported, as $\operatorname{supp}(\check{\rho} * \varphi) \subset \overline{\operatorname{supp} \check{\rho} + \operatorname{supp} \varphi}$ by PROPOSITION 7.4, and ρ, φ are compactly supported. Therefore $\check{\rho} * \varphi \in C^1_c(\mathbb{R})$ and by (*)

use $\check{\rho} * \varphi' = (\check{\rho} * \varphi)'$

$$\int_{\mathbb{R}} (\rho * u) \varphi' \, dx \overset{(*)}{=} \int_{\mathbb{R}} u (\check{\rho} * \varphi') \, dx = \int_{\mathbb{R}} u (\check{\rho} * \varphi)' \, dx$$

(As $\check{\rho} * \varphi$ is a test function and $u \in W^{1,p}(\mathbb{R})$) →

$$= - \int_{\mathbb{R}} u' (\check{\rho} * \varphi) \, dx = - \int_{\mathbb{R}} (\rho * u') \varphi \, dx$$

Use (*) with u replaced by u'

Thus $(\rho * u)' = \rho * u'$ in the weak sense. As $u' \in L^p(\mathbb{R})$, by THEOREM 7.2 we get $\rho * u' \in L^p(\mathbb{R})$, showing that $\rho * u \in W^{1,p}(\mathbb{R})$.

If ρ is not compactly supported, by COROLLARY 7.10 we can find a sequence $\{\rho_n\} \subseteq C_c(\mathbb{R})$ s.t. $\rho_n \to \rho$ strongly in $L^1(\mathbb{R})$. Note that what we proved so far holds for ρ_n, so that

$$(**) \qquad \rho_n * u \in W^{1,p}(\mathbb{R}), \qquad (\rho_n * u)' = \rho_n * u' \quad \text{weakly}, \qquad \forall\, n \in \mathbb{N}.$$

By Young's inequality we have

$$\| \rho_n * u - \rho * u \|_{L^p} \leq \| \rho_n - \rho \|_{L^1} \, \| u \|_{L^p} \to 0 \quad \text{as } n \to +\infty$$

(as $\rho_n \to \rho$ in L^1)

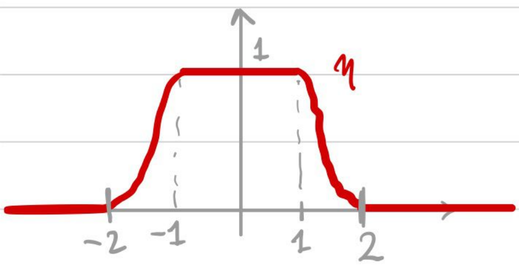
$$\| \rho_n * u' - \rho * u' \|_{L^p} \leq \| \rho_n - \rho \|_{L^1} \, \| u' \|_{L^p} \to 0 \quad \text{as } n \to +\infty$$

This means $\rho_n * u \to \rho * u$ and $(\rho_n * u)' \overset{(**)}{=} \rho_n * u' \to \rho * u'$ strongly in $L^p(\mathbb{R})$. Since $\rho * u' \in L^p(\mathbb{R})$ by THEOREM 7.2, we can invoke REMARK 7.17 to conclude that $\rho * u \in W^{1,p}(I)$, with weak derivative $(\rho * u)' = \rho * u'$. $\square$

Proof of THEOREM 7.24 Let $I \subseteq \mathbb{R}$ be open, bounded or unbounded. We need to show that for $u \in W^{1,p}(I)$ there $\exists \{u_n\} \subseteq C_c^\infty(\mathbb{R})$ s.t. $(u_n)|_I \to u$ strongly in $W^{1,p}(I)$.

First, let $\tilde{u} := Pu$ be the extension to $\mathbb{R}$ of u given by LEMMA 7.25. In particular

$$\tilde{u}|_I = u, \qquad \| \tilde{u} \|_{W^{1,p}(\mathbb{R})} \leq C \| u \|_{W^{1,p}(I)}.$$



Let $\eta \in C_c^\infty(\mathbb{R})$ be a cut-off s.t.

$$0 \leq \eta \leq 1, \quad \eta(x) = 1 \text{ for } x \in [-1,1], \quad \eta(x) = 0 \text{ for } x \in \mathbb{R} \setminus (-2,2).$$

Define $\eta_n(x) := \eta\left(\frac{x}{n}\right)$. Note that $\eta_n \to 1$ pointwise. Therefore $\eta_n \tilde{u} \to \tilde{u}$ a.e. in $\mathbb{R}$.

Since $|\eta_n \tilde{u}| \leq |\tilde{u}|$ and $\tilde{u} \in L^p(\mathbb{R})$, by Dominated Convergence (THEOREM 6.7) we get

$$(*) \quad \eta_n \tilde{u} \to \tilde{u} \text{ strongly in } L^p(\mathbb{R}).$$

Let $\rho_n \in C_c^\infty(\mathbb{R})$ be a sequence of mollifiers. Define $u_n := \eta_n \cdot (\rho_n * \tilde{u})$. Notice that $\rho_n * \tilde{u} \in C^\infty(\mathbb{R})$ by THEOREM 7.6 (indeed note that $\tilde{u} \in L^p(\mathbb{R})$ and so $u \in L^1_{loc}(\mathbb{R})$). Since $\eta_n \in C_c^\infty(\mathbb{R})$, it follows that $u_n \in C_c^\infty(\mathbb{R})$. We will show that $(u_n)_{|I} \to u$ strongly in $W^{1,p}(I)$. First note that

$$u_n - \tilde{u} = \eta_n (\rho_n * \tilde{u}) - \tilde{u} = \eta_n [\rho_n * \tilde{u} - \tilde{u}] + \eta_n \tilde{u} - \tilde{u}$$

Since $\|\eta_n\|_{L^\infty(\mathbb{R})} \leq 1$, we get

$$\|u_n - \tilde{u}\|_{L^p(\mathbb{R})} \leq \underbrace{\|\rho_n * \tilde{u} - \tilde{u}\|_{L^p(\mathbb{R})}}_{\substack{\text{This goes to } 0 \\ \text{by THEOREM 7.9}}} + \underbrace{\|\eta_n \tilde{u} - \tilde{u}\|_{L^p(\mathbb{R})}}_{\substack{\text{This goes to } 0 \\ \text{by } (*)}} \to 0 \quad \text{as } n \to +\infty$$

so $u_n \to \tilde{u}$ strongly in $L^p(\mathbb{R})$. In particular $u_n \to u$ strongly in $L^p(I)$. Also

$$u_n' = \eta_n' (\rho_n * \tilde{u}) + \eta_n (\rho_n * \tilde{u})' = \eta_n' (\rho_n * \tilde{u}) + \eta_n (\rho_n * \tilde{u}')$$

def + classical derivation of product

LEMMA 7.26 to differentiate $\rho_n * \tilde{u}$: $(\rho_n * \tilde{u})' = \rho_n * \tilde{u}'$ weakly

Note To differentiate $\rho_n * \tilde{u}$ we could also use THEOREM to get $(\rho_n * \tilde{u})' = \rho_n' * \tilde{u}$. However this term would be useless in our proof, because we need $\tilde{u}'$ to appear.

Therefore

$$\|u_n' - \tilde{u}'\|_{L^p(\mathbb{R})} = \|\eta_n' (\rho_n * \tilde{u}) + \eta_n (\rho_n * \tilde{u}') - \tilde{u}'\|_{L^p(\mathbb{R})}$$

$$\leq \|\eta_n' (\rho_n * \tilde{u})\|_{L^p(\mathbb{R})} + \|\eta_n (\rho_n * \tilde{u}') - \tilde{u}'\|_{L^p(\mathbb{R})}$$

(add subtract $\eta_n \tilde{u}'$ and use Δ inequality) $\to$

$$\leq \underbrace{\|\eta_n' (\rho_n * \tilde{u})\|_{L^p(\mathbb{R})}}_{:= I_1} + \underbrace{\|\eta_n [(\rho_n * \tilde{u}') - \tilde{u}']\|_{L^p(\mathbb{R})}}_{:= I_2} + \underbrace{\|\eta_n \tilde{u}' - \tilde{u}'\|_{L^p(\mathbb{R})}}_{:= I_3}$$

We now estimate I_1, I_2, I_3 separately.

- For I_1, notice that, as $\eta_n(x) := \eta\left(\frac{x}{n}\right)$, then $\eta_n'(x) = \frac{1}{n}\eta'\left(\frac{x}{n}\right)$.
  Setting $C := \|\eta'\|_{L^\infty(\mathbb{R})}$ we get

$$I_1 = \|\eta_n'(\rho_n * \tilde{u})\|_{L^p(\mathbb{R})}$$

$$\leq \|\eta_n'\|_{L^\infty(\mathbb{R})} \|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})}$$

$$\leq \frac{C}{n} \|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})}$$

As $\|\rho_n\|_{L^1(\mathbb{R})} = 1$ by properties of mollifiers

By Youngs' inequality we have

$$\|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})} \leq \|\rho_n\|_{L^1(\mathbb{R})} \|\tilde{u}'\|_{L^p(\mathbb{R})} = \|\tilde{u}'\|_{L^p(\mathbb{R})}$$

so that

$$I_1 \leq \frac{C}{n}\|\rho_n * \tilde{u}\|_{L^p(\mathbb{R})} \leq \frac{C}{n}\|\tilde{u}'\|_{L^p(\mathbb{R})} \to 0 \quad \text{as } n \to +\infty.$$

- For I_2,

$$I_2 = \|\eta_n\left[(\rho_n * \tilde{u}') - \tilde{u}'\right]\|_{L^p(\mathbb{R})}$$

$$\leq \|\eta_n\|_{L^\infty(\mathbb{R})} \|\rho_n * \tilde{u}' - \tilde{u}'\|_{L^p(\mathbb{R})}$$

(Since $\|\eta_n\|_{L^\infty(\mathbb{R})} = 1$)

$$= \underbrace{\|\rho_n * \tilde{u}' - \tilde{u}'\|}_{\substack{\text{This goes to } 0 \\ \text{by THEOREM 7.9,} \\ \text{as } \tilde{u}' \in L^1_{loc}(\mathbb{R})}}{}_{L^p(\mathbb{R})} \to 0 \quad \text{as } n \to +\infty.$$

- For I_3: Recall that $\eta_n \to 1$ pointwise in $\mathbb{R}$. Thus $\eta_n \tilde{u}' \to \tilde{u}'$ a.e. in $\mathbb{R}$. Also $|\eta_n \tilde{u}'| \leq |\tilde{u}'|$ as $|\eta_n| \leq 1$. Then we can apply DOMINATED CONVERGENCE to get

$$\eta_n \tilde{u}' \to \tilde{u}' \quad \text{strongly in } L^p(\mathbb{R}),$$

which implies

$$I_3 = \| \eta_n \tilde{u}' - \tilde{u}' \|_{L^p(\mathbb{R})} \longrightarrow 0 \quad \text{as } n \to +\infty$$

In total, we just proved that

$$\| u_n' - \tilde{u}' \|_{L^p(\mathbb{R})} \leq I_1 + I_2 + I_3 \to 0 \quad \text{as } n \to +\infty,$$

that is, $u_n' \to \tilde{u}'$ strongly in $L^p(\mathbb{R})$. In particular,

$$(**) \quad u_n' \to \tilde{u}'|_I \quad \text{strongly in } L^p(I)$$

Now recall that we had

$$(***) \quad u_n \to u \quad \text{strongly in } L^p(I)$$

Note that $\{u_n\} \subseteq W^{1,p}(I)$, as $\{u_n\} \subseteq C_c^\infty(\mathbb{R})$. Therefore, as $(**)$, $(***)$ hold we can apply REMARK 7.17 and conclude

$$u_n \to u \quad \text{strongly in } W^{1,p}(I),$$

ending the proof. $\square$