LESSON 7 - 28 APRIL 2021

DEFINITION 7.3 (SUPPORT)

Let $u : \mathbb{R}^d \to \mathbb{R}$. Let $\{w_i\}_{i \in I}$ be the family of all open sets in $\mathbb{R}^d$ s.t.

$$u = 0 \quad \text{a.e. on } w_i \, , \quad \forall i \in I.$$

We define the support of u as

$$\operatorname{Supp} u := \mathbb{R}^d \setminus \bigcup_{i \in I} w_i \, .$$

REMARK The above definition makes sense, since it is possible to show that:

1. $u = 0$ a.e. on $\bigcup_{i \in I} w_i$

2. If $u_1 = u_2$ a.e. on $\mathbb{R}^d$, then $\operatorname{supp} u_1 = \operatorname{supp} u_2$.

3. If u is continuous, then DEFINITION 7.3 coincides with the classical one, i.e.,

$$\operatorname{Supp} u = \mathbb{R}^d \setminus \bigcup_{i \in I} w_i = \overline{\{ x \in \mathbb{R} \mid u(x) \neq 0 \}}$$

EXAMPLE Again consider $u := \chi_{\mathbb{Q}}$. As $\mathcal{L}(\mathbb{Q}) = 0$, we know that $u = 0$ a.e. on $\mathbb{R}$. Therefore $u = 0$ a.e. on w, for all $w \subseteq \mathbb{R}$ open. It follows that $\operatorname{Supp} u = \mathbb{R} \setminus \bigcup_{i \in I} w_i = \mathbb{R} \setminus \mathbb{R} = \emptyset$. This coincides with $\operatorname{supp} v$, where $v \equiv 0$.

The support of a convolution can be estimated in the following way:

PROPOSITION 7.4

Let $u \in L^1(\mathbb{R})$, $v \in L^p(\mathbb{R})$ for some $1 \leq p \leq +\infty$. Then

$$(*) \quad \operatorname{Supp}(u * v) \subset \overline{\operatorname{Supp} u + \operatorname{supp} v}$$

The sum in $(*)$ is defined as $E + F := \{ x + y \, , \, x \in E, y \in F \}$, where $E, F \subseteq \mathbb{R}$ subsets.

(The proof of PROPOSITION 7.4 will be left as an Exercise in the EX. COURSE)

The main point of introducing convolutions is that they have a smoothing effect. To make this statement rigorous we need the definition of LOCAL INTEGRABILITY.

DEFINITION 7.5 Let $\Omega \subset \mathbb{R}^d$ be open. Let $1 \leq p \leq +\infty$. We say that $u : \Omega \to \mathbb{R}$ is **LOCALLY INTEGRABLE** on Ω, if

$$u \chi_K \in L^p(\Omega) \quad \text{for all} \quad K \subset \Omega, \; K \text{ compact.}$$

The space of locally integrable functions on Ω is denoted by $L^p_{loc}(\Omega)$

REMARK We have that $L^p_{loc}(\Omega) \subset L^1_{loc}(\Omega)$, for all $\Omega \subseteq \mathbb{R}^d$ open

(This is not true for $L^p(\Omega)$ with Ω unbounded)

THEOREM 7.6 (Smoothing via convolutions)

(a) Let $u \in C_c(\mathbb{R})$, $v \in L^1_{loc}(\mathbb{R})$. Then $(u * v)(x)$ is well-defined $\forall x \in \mathbb{R}$ and

$$u * v \in C(\mathbb{R})$$

(b) Let $k \geq 1$, $u \in C^k_c(\mathbb{R})$, $v \in L^1_{loc}(\mathbb{R})$. Then $(u * v) \in C^k(\mathbb{R})$ and

$$\frac{d^k}{dx^k}[u * v] = u^{(k)} * v$$

In particular, if $u \in C^\infty_c(\mathbb{R})$ and $v \in L^1_{loc}(\mathbb{R})$, then $(u * v) \in C^\infty(\mathbb{R})$.

(The proof of this Theorem will be left as an ex. for the Exercises Course)

DEFINITION 7.7 (MOLLIFIERS) A sequence of **MOLLIFIERS** is any sequence $\rho_n : \mathbb{R} \to \mathbb{R}$ s.t.

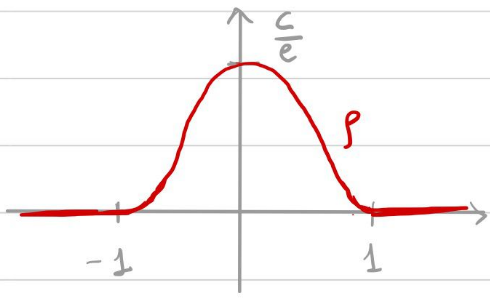
$$\rho_n \in C^\infty_c(\mathbb{R}), \quad \rho_n \geq 0, \quad \operatorname{supp} \rho_n \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right], \quad \int_{\mathbb{R}} \rho_n(x)\,dx = 1, \quad \forall n \in \mathbb{N}.$$

The most commonly used sequence of mollifiers is defined as follows:

EXAMPLE (STANDARD MOLLIFIERS)

Let $\rho: \mathbb{R} \to \mathbb{R}$ be as in REMARK 3.2, i.e.,

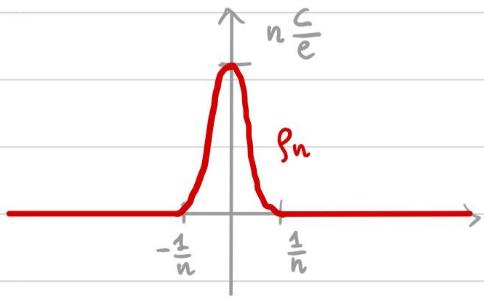
$$\rho(x) := \begin{cases} C \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$



where $C \in \mathbb{R}$ is $C := \left(\int_{\mathbb{R}} \rho(x)\,dx \right)^{-1}$. Then $\rho \in C_c^\infty(\mathbb{R})$, $\rho \geq 0$, $\operatorname{supp} \rho \subset [-1, 1]$, $\int_{\mathbb{R}} \rho = 1$.

In particular $\rho_n: \mathbb{R} \to \mathbb{R}$ defined by

$$\boxed{\rho_n(x) := n\,\rho(nx)}$$



is a sequence of mollifiers.

PROPOSITION 7.8 Let $u \in C(\mathbb{R})$ and $\{\rho_n\}$ be a sequence of mollifiers. Then $\rho_n * u \to u$ uniformly on compact sets, i.e. for each $K \subset \mathbb{R}$ compact we have

$$\lim_{n \to +\infty} \max_{x \in K} \left| (\rho_n * u)(x) - u(x) \right| = 0.$$

(Also the proof of this is left as exercise for the EX. COURSE)

THEOREM 7.9 Let $1 \leq p < +\infty$, $u \in L^p(\mathbb{R})$, $\{\rho_n\}$ a sequence of mollifiers. Then

$$\rho_n * u \to u \quad \text{STRONGLY in } L^p(\mathbb{R}).$$

Proof Let $\varepsilon > 0$. By THEOREM 6.16 $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$. Thus $\exists\, \tilde{u} \in C_c(\mathbb{R})$ such that

$$(*) \qquad \| u - \tilde{u} \|_p < \varepsilon .$$

By PROPOSITION 7.8 we have that $\rho_n * \tilde{u} \to \tilde{u}$ uniformly on compact sets. Moreover PROPOSITION 7.4 says that

by def of ρ_n

$$\operatorname{supp}(\rho_n * \tilde{u}) \subset \overline{\operatorname{supp}\rho_n + \operatorname{supp}\tilde{u}} \subseteq \left[-\tfrac{1}{n}, \tfrac{1}{n}\right] + \operatorname{supp}\tilde{u} \subseteq [-1,1] + \operatorname{supp}\tilde{u}$$

In particular

$$\begin{aligned} \operatorname{supp}(\rho_n * \tilde{u} - \tilde{u}) &\subset \left(\operatorname{supp}(\rho_n * \tilde{u}) \cup \operatorname{supp}\tilde{u}\right) \\ (**) \qquad &\subset \left(\{[-1,1] + \operatorname{supp}\tilde{u}\} \cup \operatorname{supp}\tilde{u}\right) \\ &= [-1,1] + \operatorname{supp}\tilde{u} =: K \end{aligned}$$

and K is compact, as $\tilde{u}$ is compactly supported. Then

$$\begin{aligned} \int_{\mathbb{R}} |(\rho_n * \tilde{u})(x) - \tilde{u}(x)|^p \, dx &\overset{(**)}{=} \int_K |(\rho_n * \tilde{u})(x) - \tilde{u}(x)|^p \, dx \\ &\leq \max_{x \in K} |(\rho_n * \tilde{u})(x) - \tilde{u}(x)|^p \cdot |K| \end{aligned}$$

and the RHS goes to zero as $n \to +\infty$, since $\rho_n * \tilde{u} \to \tilde{u}$ uniformly on compact sets, and $|K| < +\infty$ being K bounded (as it is compact). Thus

$$(***) \qquad \rho_n * \tilde{u} \to \tilde{u} \quad \text{STRONGLY in } L^p(\mathbb{R})$$

Now notice that $\rho_n \in L^1(\mathbb{R})$ with $\|\rho_n\|_1 = 1$ by definition. Moreover $(u-\tilde{u}) \in L^p(\mathbb{R})$. Therefore by YOUNG INEQUALITY (THEOREM 7.2)

$$\text{Y} \qquad \|\rho_n * (u-\tilde{u})\|_p \leq \|\rho_n\|_1 \|u-\tilde{u}\|_p = \|u-\tilde{u}\|_p$$

Finally, by adding and subtracting we can estimate

$$\begin{aligned}
\|\rho_n * u - u\|_p &\leq \|\rho_n * (u-\tilde{u})\|_p + \|\rho_n * \tilde{u} - \tilde{u}\|_p + \|u-\tilde{u}\|_p \\
&\overset{\text{Y}}{\leq} \|u-\tilde{u}\|_p + \|\rho_n * \tilde{u} - \tilde{u}\|_p + \|u-\tilde{u}\|_p \\
&= 2\|u-\tilde{u}\|_p + \|\rho_n * \tilde{u} - \tilde{u}\|_p
\end{aligned}$$

Recalling that $\|u-\tilde{u}\|_p < \varepsilon$ and $\|\rho_n * \tilde{u} - \tilde{u}\|_p \to 0$ by (***), we get

$$0 \leq \limsup_{n\to+\infty} \|\rho_n * u - u\|_p \leq 2\varepsilon$$

As $\varepsilon > 0$ was arbitrary, we conclude $\|\rho_n * u - u\|_p \to 0$. $\square$

COROLLARY 7.10 Let $I \subset \mathbb{R}$ be open, $1 \leq p < +\infty$. Then $C_c^\infty(I)$ is dense in $L^p(I)$.

FLCV and DBR in L^p

We are now ready to prove L^p versions of the FLCV and DBR Lemma.

LEMMA 7.11 (FLCV in L^p)

Let $I \subset \mathbb{R}$ be open. Suppose $u \in L^1_{loc}(I)$ is s.t.

$$\int_I u v \, dx = 0 , \qquad \forall v \in C^\infty_c(I) .$$

Then $u = 0$ a.e. on I.

Proof Let $\psi \in L^\infty(\mathbb{R})$ be s.t. supp ψ is compact and contained in I. Let $\psi_n := \rho_n * \psi$, with ρ_n mollifier. Then

$$\operatorname{supp} \psi_n \subset \overline{\operatorname{supp} \rho_n + \operatorname{supp} \psi} \overset{\text{by def of } \rho_n}{=} \overline{\left[-\tfrac{1}{n}, \tfrac{1}{n}\right] + \operatorname{supp} \psi}$$

by PROPOSITION 7.4. Then there $\exists \, N \in \mathbb{N}$ s.t. supp $\psi_n \subset I$ for all $n \geq N$. Moreover $\psi_n \in C^\infty(\mathbb{R})$ by THEOREM 7.6. Thus $\psi_n \in C^\infty_c(I)$ for all $n \geq N$. By assumption we get

$$(*) \qquad \int_I u \psi_n \, dx = 0 , \qquad \forall n \geq N .$$

Notice that $\psi \in L^1(\mathbb{R})$, being compactly supported and in $L^\infty(\mathbb{R})$. Then by THEOREM 7.9 we have $\psi_n \to \psi$ strongly in $L^1(\mathbb{R})$. Therefore, up to subsequencies (not relabelled), we have $\psi_n \to \psi$ pointwise a.e. in $\mathbb{R}$ (see PROPOSITION 6.13). Also, by YOUNG'S INEQUALITY, we get $\| \psi_n \|_{L^\infty(\mathbb{R})} \leq \| \rho_n \|_{L^1(\mathbb{R})} \| \psi \|_{L^\infty(\mathbb{R})} = \| \psi \|_{L^\infty(\mathbb{R})}$, as $\| \rho_n \|_{L^1(\mathbb{R})} = 1$ by definition. Therefore

$$u \psi_n \to u \psi \ \text{ a.e. in } I , \qquad |u \psi_n| \leq \| \psi \|_{L^\infty(\mathbb{R})} |u| \in L^1(I)$$

Therefore we can invoke DOMINATED CONVERGENCE (THEOREM 6.7) to conclude

$$\int_I u \psi_n \, dx \to \int_I u \psi \, dx \qquad \text{as } n \to +\infty .$$

However $\int_I u \psi_n \, dx = 0$ for n sufficiently large, by (*), and so

$$\int_I u \psi \, dx = 0 .$$

Therefore we have proven that

$$(**) \quad \int_I u \psi \, dx = 0 , \quad \forall \psi \in L^\infty(I) \text{ s.t. } \operatorname{supp}\psi \text{ is compact, } \operatorname{supp}\psi \subset I$$

To obtain our thesis we now choose a function ψ in (**) in a clever way: Let $K \subset I$ be compact and define

$$\tilde{\psi}(x) := \begin{cases} \operatorname{sign} u(x) & \text{if } x \in K \\ 0 & \text{if } x \in \mathbb{R} \setminus K \end{cases} \qquad \left(\operatorname{sign} a = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases} \right)$$

Therefore $\tilde{\psi} \in L^\infty(\mathbb{R})$ and $\operatorname{supp}\tilde{\psi} \subset K \subset I$. Then, from (**),

$$0 \overset{(**)}{=} \int_I u \tilde{\psi} \, dx = \int_K u \operatorname{sign} u \, dx = \int_K |u| \, dx .$$

Then $u = 0$ a.e. on K by the properties of the Lebesgue integral (see the REMARK after THEOREM 6.4). As K is arbitrary, we conclude $u = 0$ a.e. on I. □

LEMMA 7.12 (DBR in L^p)

Let $I = (a,b)$, possibly unbounded. Let $u \in L^1_{loc}(I)$ be such that

$$\int_I u v \, dx = 0 , \quad \forall v \in C_c(I) \text{ such that } \int_I v \, dx = 0 .$$

Then $u = C$ a.e. on I, for some constant $C \in \mathbb{R}$.

Proof Let $\psi \in C_c(I)$ s.t. $\int_I \psi \, dx = 1$ (e.g. the bump function of REMARK 3.2, suitably rescaled).
Let $w \in C_c(I)$ be arbitrary and set

$$h(x) := w(x) - \left(\int_I w \, dx \right) \psi(x)$$

Then $h \in C_c(I)$, since $w, \psi \in C_c(I)$. Also $\int_I h \, dx = 0$ as $\int_I \psi \, dx = 1$. Then by assumption $\int_I u h \, dx = 0$. Thus

$$0 = \int_I u h \, dx = \int_I u w \, dx - \int_I w \, dx \cdot \int_I u \psi \, dx = \int_I (u - c) \, w \, dx \Rightarrow \int_I (u-c) \, w \, dx = 0$$

with $c := \int_I u \psi \, dx$. As c does not depend on w, and $w \in C_c(I)$ is arbitrary, by FLCV LEMMA 7.11 we conclude $u = c$ a.e. on I. $\square$

Similarly to the regular DBR Lemma, we have an alternative version:

LEMMA 7.13 (DBR in L^p - Alternative version)

Let $I = (a,b)$, possibly unbounded. Let $u \in L^1_{loc}(I)$ be such that

$$\int_I u \dot{v} \, dx = 0, \quad \forall v \in C^1_c(I).$$

Then $u = c$ a.e. on I, for some constant $c \in \mathbb{R}$.

Proof Let $w \in C_c(I)$ be such that $\int_I w \, dx = 0$. Set $W(x) := \int_a^x w(t) \, dt$. Then $W \in C^1(I)$ by the Fundamental Theorem of Calculus and $\dot{W}(x) = w(x)$. Moreover W is compactly supported in I, since w is compactly supported in I and $\int_I w \, dx = 0$. Thus $W \in C^1_c(I)$ and by assumption we get $\int_I u \dot{W} \, dx = 0$. As $\dot{W} = w$, we get

$$\int_I u w \, dx = 0, \quad \forall w \in C_c(I) \text{ s.t. } \int_I w \, dx = 0.$$

We can now apply LEMMA 7.12 and conclude $u = c$ a.e. on I, for some $c \in \mathbb{R}$. $\square$

SOBOLEV SPACES

MOTIVATION Let $I=(a,b)\subseteq\mathbb{R}$. If $u\in C^1_{pw}[a,b]$ then "morally" $u\in C^1[a,b]$ with the exception of a few points, in which u' is discontinuous.

The idea is that, if we integrate u', the Lebesgue integral does not see those exceptional points, as that set has zero measure. Now, if $\varphi\in C^1_c(a,b)$, we get, integrating by parts,

$$\int_a^b u\varphi'\,dx = \underbrace{u\varphi\Big|_a^b}_{\substack{=0 \text{ as}\\ \varphi(a)=\varphi(b)=0}} - \int_a^b u'\varphi\,dx$$

Thus for $u\in C^1_{pw}[a,b]$ and $\varphi\in C^1_c(a,b)$ we get

$$\int_a^b u\varphi'\,dx = -\int_a^b u'\varphi\,dx$$

Note that the above expression makes sense also if u,u' only belong to $L^1(a,b)$. This motivates the following definition.

DEFINITION 7.14 (Sobolev Space)

Let $I=(a,b)$ be an interval, possibly unbounded. Let $1\le p\le+\infty$. The SOBOLEV SPACE $W^{1,p}(I)$ is defined as

$$W^{1,p}(I):=\Big\{u\in L^p(I)\ \Big|\ \exists\, g\in L^p(I) \text{ s.t. } \underbrace{\int_I u\dot\varphi\,dx = -\int_I g\varphi\,dx}_{\text{"INTEGRATION BY PARTS"}},\ \forall \varphi\in C^1_c(I)\Big\}.$$

REMARK If $u\in W^{1,p}(I)$ then the function g is UNIQUE

Proof Assume $g, h \in L^p(I)$ both satisfy the "integration by parts" formula, i.e.,

$$\int_I u\dot{\varphi}\,dx = -\int_I g\varphi \quad , \quad \forall \varphi \in C_c^1(I)$$

$$\int_I u\dot{\varphi}\,dx = -\int_I h\varphi \quad , \quad \forall \varphi \in C_c^1(I)$$

Then $\int_I (g-h)\varphi\,dx = 0$, $\forall \varphi \in C_c^1(I)$. By FLCV LEMMA 7.11

we get $g=h$ a.e. on I. Thus g and h are the same L^p function, as they belong to the same equivalence class.

NOTATION ① If $u \in W^{1,p}(I)$ and g is the function from the definition, we denote

$$\dot{u} := g$$

and call $\dot{u}$ the WEAK DERIVATIVE of u.

② For $p=2$ we denote $H^1(I) := W^{1,2}(I)$.

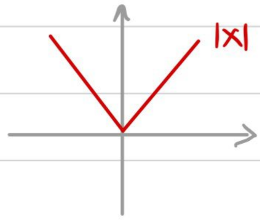
EXAMPLE 7.15 ① If $u \in C^1(I) \cap L^p(I)$ then $u \in W^{1,p}(I)$ and the weak derivative coincides with the classical derivative

② If I is bounded then $C^1(\bar{I}) \subset W^{1,p}(I)$ for all $1 \leq p \leq +\infty$

③ If $u \in C^1_{pw}(I) \cap L^p(I)$ then $u \in W^{1,p}(I)$ and the weak derivative coincides with the classical derivative in the points of differentiability of u (Note that this makes sense, since the weak derivative needs only to be defined almost everywhere, and the set of points where u is not differentiable is finite $\Rightarrow$ it has zero measure)

For example consider $I = (-1, 1)$, $u(x) := |x|$. Then $u \in C^1_{pw}(I)$ with

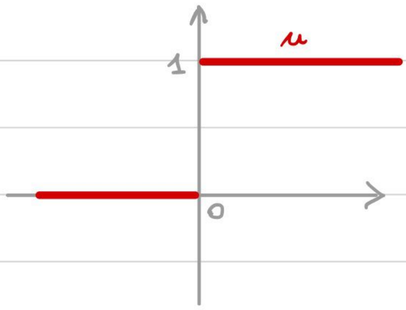
$$\dot{u}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



It is easy to check that this is the weak derivative of u.

④ Functions with JUMPS do not belong to $W^{1,p}(I)$. For example consider $I := (-1, 1)$ and

$$u(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$



The pointwise derivative of u is the function $\dot{u}(x) = 0$ for all $x \in \mathbb{R} \setminus \{0\}$. Thus we also have that $\dot{u} \in L^p(I)$. However it is easy to show that $\dot{u}$ is NOT the weak derivative of u. Moreover one can show that u does not admit any weak derivative, i.e., $u \notin W^{1,p}(I)$, for any $1 \le p \le +\infty$.

NOTATION

① The space $W^{1,p}(I)$ is equipped with the NORM

$$\boxed{\|u\|_{W^{1,p}} := \|u\|_{L^p} + \|\dot{u}\|_{L^p} .}$$

② If $1 \le p < +\infty$, $W^{1,p}(I)$ can be equipped with the EQUIVALENT NORM

$$\boxed{\|u\|_{W^{1,p}} := \left(\|u\|^p_{L^p} + \|\dot{u}\|^p_{L^p} \right)^{1/p}.}$$

③ The space $H^1(I)$ can be equipped with the INNER PRODUCT

$$\boxed{\langle u, v \rangle_{H^1} := \langle u, v \rangle_{L^2} + \langle \dot{u}, \dot{v} \rangle_{L^2}}$$

The induced norm is

$$\boxed{\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|\dot{u}\|_{L^2}^2 \right)^{1/2}}$$

[Checking the above statements is straightforward, using that $\|u\|_{L^p}$ is a norm on L^p and that $\langle u, v \rangle_{L^2}$ is an inner product on L^2]

PROPOSITION 7.16 Let $I \subseteq \mathbb{R}$ be open, bounded or unbounded. Then:

1. $W^{1,p}(I)$ is a BANACH SPACE for $1 \leq p \leq +\infty$.

2. $W^{1,p}(I)$ is REFLEXIVE for $1 < p < +\infty$.

3. $W^{1,p}(I)$ is SEPARABLE for $1 \leq p < +\infty$.

4. $H^1(I)$ is a SEPARABLE HILBERT space.

Proof ① We need to prove that $W^{1,p}(I)$ is complete. So let $\{u_n\} \subseteq W^{1,p}(I)$ be a Cauchy sequence. As

$$\|u\|_{L^p} \leq \|u\|_{W^{1,p}} \quad \text{and} \quad \|\dot{u}\|_{L^p} \leq \|u\|_{W^{1,p}} \quad , \forall u \in W^{1,p}(I)$$

we have that $\{u_n\}, \{\dot{u}_n\}$ are Cauchy sequences in $L^p(I)$. As $L^p(I)$ is complete, there $\exists\ u, g \in L^p(I)$ s.t.

$$(*) \qquad u_n \to u \quad , \quad \dot{u}_n \to g \quad \text{strongly in } L^p(I).$$

By definition of $W^{1,p}$ we have

$$(**) \qquad \int_I u_n \dot{\varphi}\, dx = - \int_I \dot{u}_n \varphi\, dx \quad , \quad \forall \varphi \in C_c^1(I) \quad , \quad \forall n \in \mathbb{N}$$

This is $L^p(I)^*$.
See THEOREM 6.14

As $u_n \to u$ strongly, then $u_n \rightharpoonup u$ weakly in $L^p(I)$. Since $C_c(I) \subset L^{p'}(I)$ we get

$$\int_I u_n \dot{\varphi}\, dx \to \int_I u \varphi\, dx \qquad \text{as } n \to +\infty$$

Similarly

$$\int_I \dot{u}_n \varphi\, dx \to \int_I g \varphi\, dx \qquad \text{as } n \to +\infty$$

Then we can pass to the limit in (**) and obtain

$$\int_I u \dot{\varphi} = -\int_I g \varphi\, dx \quad , \quad \forall \varphi \in C_c^1(I).$$

This shows $u \in W^{1,p}(I)$ with weak derivative $\dot{u} = g$. Then by (*) we conclude $\|u_n - u\|_{W^{1,p}} \to 0$, showing completeness.

(2) Recall that $L^p(I)$ is REFLEXIVE for $1 < p < +\infty$ (THEOREM 6.14). Then it is easy to check that $E := L^p(I) \times L^p(I)$ is reflexive. Let

$$T : W^{1,p}(I) \to E$$
$$u \mapsto (u, \dot{u})$$

One can check that T is an isometry. Since $W^{1,p}(I)$ is Banach, it follows that $T(W^{1,p}(I)) \subseteq E$ is a closed subspace. Since closed subspaces of reflexive spaces are reflexive, we conclude.

(3) $L^p(I)$ is separable for all $1 \leq p < +\infty$ (THEOREM 6.15). Thus $E := L^p(I) \times L^p(I)$ is separable (immediate check). Consider T as above. As any SUBSET of a separable space is separable, from the inclusion $T(W^{1,p}(I)) \subseteq E$ we conclude.

(4) Follows from (1) and (3). $\square$

In the above proof, point 1, we showed a general fact which is worthy of its own numbered Remark.

REMARK 7.17 Let $\{u_n\} \subseteq W^{1,p}(I)$ be such that

$$\begin{cases} u_n \to u & \text{in } L^p(I) \\ \dot{u}_n \to g & \text{in } L^p(I) \end{cases}$$

Then $u_n \to u$ in $W^{1,p}(I)$ and $u \in W^{1,p}(I)$, with $\dot{u} = g$.

A similar Remark holds also for weak convergence.

REMARK 7.18 Let $\{u_n\} \subseteq H^1(I)$ be such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } L^2(I) \\ \dot{u}_n \rightharpoonup g & \text{weakly in } L^2(I) \end{cases}$$

Then $u \in H^1(I)$ with $\dot{u} = g$ in the weak sense and $u_n \rightharpoonup u$ in $H^1(I)$.

Proof As $u_n \in H^1(I)$ then

$$(*) \quad \int_I u_n \dot{\varphi}\, dx = - \int_I \dot{u}_n \varphi\, dx \quad , \quad \forall \varphi \in C_c^1(I) \quad , \quad \forall n \in \mathbb{N}.$$

As $L^2(I)$ is Hilbert, we have that $L^2(I)^* = L^2(I)$. Since $C_c^1(I) \subseteq L^2(I)$, by the weak convergence $u_n \rightharpoonup u$, we get

$$\int_I u_n \dot{\varphi}\, dx \to \int_I u \dot{\varphi}\, dx .$$

Similarly, as $\dot{u}_n \rightharpoonup g$, we get

$$\int_I \dot{u}_n \varphi\, dx \longrightarrow \int_I g \varphi\, dx .$$

Then we can pass to the limit in ⊛ and get that $\dot{u} = g$ in the weak sense. As $g \in L^2(I)$, we get $u \in H^1(I)$. If $v \in H^1(I)$ we get

$$\langle u_n, v \rangle_{H^1} = \langle u_n, v \rangle_{L^2} + \langle \dot{u}_n, \dot{v} \rangle_{L^2} \to \langle u, v \rangle_{L^2} + \langle \dot{u}, \dot{v} \rangle_{L^2} = \langle u, v \rangle_{H^1}$$

showing that $u_n \rightharpoonup u$ weakly in $H^1(I)$. $\square$

We now prove one of the main results on 1-dimensional Sobolev functions, namely, that they are CONTINUOUS and they are PRIMITIVES of L^p functions.

THEOREM 7.19 Let $I = (a,b)$ be bounded or unbounded, and $1 \le p \le +\infty$. Let $u \in W^{1,p}(I)$. Then there $\exists\, \tilde{u} \in C(I)$ s.t.

$$u = \tilde{u} \quad \text{a.e. on } I$$

and

$$\tilde{u}(x) - \tilde{u}(y) = \int_y^x \dot{u}(t)\, dt \quad , \quad \forall\, x, y \in I.$$

NOTE Theorem 7.19 is saying that if $u \in W^{1,p}(I)$ then $\exists\, \tilde{u}$ continuous in the same equivalence class of u. We call $\tilde{u}$ the CONTINUOUS REPRESENTATIVE of u, and in the future we just denote it by u.

During the proof of THEOREM 7.19 we need the following lemma.

LEMMA 7.20 $I = (a,b)$, $g \in L^1_{loc}(I)$. Fix $y_0 \in I$ and define

$$u(x) := \int_{y_0}^x g(t)\, dt \quad , \quad \forall x \in I.$$

Then $u \in C(I)$ and $\dot{u} = g$ in the weak sense.