5. SUFFICIENT CONDITIONS FOR MINIMALITY

LESSON 6
21 APRIL 2021

So far we have shown that solutions to a minimization problem for integral functionals also solve the associated EULER-LAGRANGE EQUATION.

QUESTION: Are solutions to (ELE) minimizers? If YES, how do we prove it?

WARNING: This is in general not true: we can have $\bar{u}$ sol. to (ELE) but not minimizer

To answer the above, we will analyze 4 methods:

(1) CONVEXITY
(2) TRIVIAL LEMMA
} NOW

(3) CALIBRATIONS
(4) WEIERSTRASS FIELDS
} LATER in the course (if we have time!)

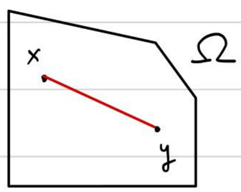
(1) CONVEXITY

If the Lagrangian $L = L(x,s,p)$ is convex in s,p, we will prove that solutions to (ELE) are minimizers.

DEFINITION 5.1

Let $\Omega \subseteq \mathbb{R}^d$. We say that Ω is convex if

$$\lambda x + (1-\lambda) y \in \Omega, \quad \forall\, x,y \in \Omega, \ \lambda \in [0,1].$$

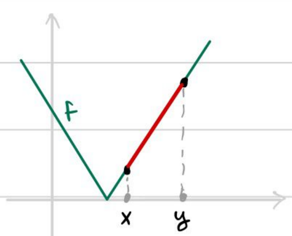


CONVEX SET Ω

Let $f: \Omega \to \mathbb{R}$, with Ω convex. We say that f is convex if

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

for all $x,y \in \Omega$, $\lambda \in [0,1]$.

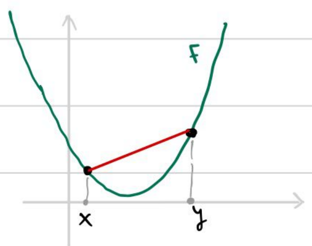


f CONVEX BUT NOT STRICTLY

We say that f is STRICTLY CONVEX if

$$f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y),$$

for all $x,y \in \Omega$, s.t. $x \neq y$ and $\lambda \in (0,1)$.



f STRICTLY CONVEX

For regular convex functions the following result holds:

THEOREM 5.2 Let $\Omega \subseteq \mathbb{R}^d$ be open convex, $f : \Omega \to \mathbb{R}$, $f \in C^1(\Omega)$. Then

1) f is convex iff

(f above tangent planes) $\rightsquigarrow$ $f(y) \geq f(x) + \nabla f(x) \cdot (y-x) \quad , \quad \forall x, y \in \Omega$

2) f is strictly convex iff

$$f(y) > f(x) + \nabla f(x) \cdot (y-x) \quad , \forall x, y \in \Omega \, , \, x \neq y$$

Assume in addition that $f \in C^2(\Omega)$

3) f is convex iff the HESSIAN $\nabla^2 f$ is POSITIVE SEMI-DEFINITE, i.e.,

$$y^T \nabla^2 f(x) \, y \geq 0, \quad \forall x \in \Omega, \, y \in \mathbb{R}^d$$

4) Assume $\nabla^2 f$ is POSITIVE DEFINITE, i.e.,

$$y^T \nabla^2 f(x) \, y > 0, \quad \forall x \in \Omega \, , \, y \in \mathbb{R}^d \setminus \{0\}$$

Then f is strictly convex.

(The proof is standard, from analysis courses. See B. DACOROGNA - "INTRODUCTION TO THE CALCULUS OF VARIATIONS", IMPERIAL COLLEGE PRESS, 2004 - THEOREM 1.5)

WARNING: The converse of (4) does not hold.

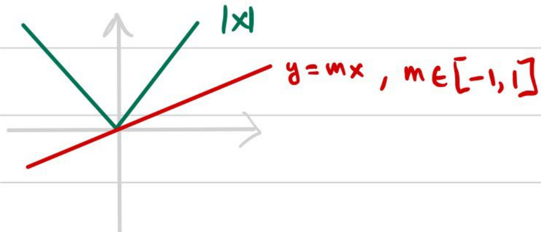
If instead we have no regularity, then we get:

THEOREM 5.3 Let $\Omega \subseteq \mathbb{R}^d$ be convex, and $f:\Omega \to \mathbb{R}$ convex. Let $\bar{x} \in \Omega$. Then $\exists\, m \in \mathbb{R}^d$ s.t.

$$f(y) \geq f(\bar{x}) + m \cdot (y - \bar{x}), \quad \forall y \in \Omega$$

(Proof is omitted. This result is saying that if f convex then $\partial f(\bar{x}) \neq \emptyset$, i.e., the SUBDIFFERENTIAL of f at $\bar{x}$ is non-empty. For a proof see R.T. ROCKAFELLAR – "CONVEX ANALYSIS", PRINCETON UNIVERSITY PRESS, 1970 – THEOREM 23.4)

NOTE: For $f:[a,b] \to \mathbb{R}$ one can take $m \in [f'_-(\bar{x}), f'_+(\bar{x})]$ left and right derivatives.



APPLICATION TO COV

Let $X := \{u \in C^1[a,b] \mid u(a) = \alpha, u(b) = \beta\}$, $V := \{v \in C^1[a,b] \mid v(a) = v(b) = 0\}$.
Let $L: [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x,s,p)$ and define $F: X \to \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx.$$

THEOREM 5.4 Suppose $L \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$ and let $\bar{u} \in X$ be a solution to ELE in INTEGRAL FORM, i.e.,

$$(*) \quad \int_a^b L_s(x, \bar{u}, \bar{u}')\, v + L_p(x, \bar{u}, \bar{u}')\, v' \; dx = 0, \quad \forall v \in V.$$

(1) If $(s,p) \mapsto L(x,s,p)$ is CONVEX for all $x \in [a,b]$ fixed, then $\bar{u}$ is a minimizer for F in X.

(2) If $(s,p) \mapsto L(x,s,p)$ is STRICTLY CONVEX for all $x \in [a,b]$, then $\bar{u}$ is the unique minimizer of F in X.

NOTE: If u solves ELE in DIFFERENTIAL FORM then it solves ELE in INTEGRAL FORM

<u>Proof</u> (1) Let $w \in X$ be arbitrary and set $v := w - \bar{u}$.
Then $v \in V$ i.e. $v(a) = v(b) = 0$. We have

$$F(w) = F(\bar{u} + v) = \int_a^b L(x, \bar{u} + v, \bar{u}' + v')\, dx$$

As L is C^1 and is convex in s, p, we can apply Theorem 5.2 and obtain

$$L(x, s+\tilde{s}, p+\tilde{p}) \geq L(x, s, p) + L_s(x,s,p)\,\tilde{s} + L_p(x,s,p)\,\tilde{p}, \quad \begin{matrix} \forall s, \tilde{s}, p, \tilde{p} \in \mathbb{R} \\ \forall x \in [a,b] \end{matrix}$$

Apply the above with $s = \bar{u}$, $\tilde{s} = v$, $p = \bar{u}'$, $\tilde{p} = v'$,

$$F(w) = \int_a^b L(x, \bar{u}+v, \bar{u}'+v')\, dx \geq$$

$$\geq \int_a^b L(x, \bar{u}, \bar{u}')\, dx + \underbrace{\int_a^b L_s(x, \bar{u}, \bar{u}')\, v + L_p(x, \bar{u}, \bar{u}')\, v'\, dx}_{=0 \text{ by } (*), \text{ since } v \in V}$$

$$= F(\bar{u})$$

showing that $\bar{u}$ minimizes F over X.

(2) Assume $\bar{u}$ and $\hat{u}$ both minimize F over X. Set $m := \min\{F(u) \mid u \in X\}$.
Therefore $F(\hat{u}) = F(\bar{u}) = m$, and also $F(u) \geq m$, $\forall u \in X$.

Define $w := \frac{1}{2}\bar{u} + \frac{1}{2}\hat{u}$, so $w \in X$. By convexity of L (just using the definition)

$$L(x, w, w') = L\left(x, \tfrac{1}{2}\bar{u} + \tfrac{1}{2}\hat{u}, \tfrac{1}{2}\bar{u}' + \tfrac{1}{2}\hat{u}'\right)$$

$$\leq \tfrac{1}{2} L(x, \bar{u}, \bar{u}') + \tfrac{1}{2} L(x, \hat{u}, \hat{u}')$$

Integrating the above inequality we obtain

$$m \leq F(w) = \int_a^b L(x, w, w')\,dx \leq$$

Since $w \in X$

$$\leq \frac{1}{2}\int_a^b L(x, \bar{u}, \bar{u}')\,dx + \frac{1}{2}\int_a^b L(x, \hat{u}, \hat{u}')\,dx =$$

$$= \frac{1}{2} F(\bar{u}) + \frac{1}{2} F(\hat{u})$$

$$= \frac{1}{2} m + \frac{1}{2} m = m$$

Thus, all the inequalities in the above chain are actually equalities, and we get

$$(*) \quad \int_a^b \left\{ \frac{1}{2} L(x, \bar{u}, \bar{u}') + \frac{1}{2} L(x, \hat{u}, \hat{u}') - L\left(x, \frac{1}{2}\bar{u} + \frac{1}{2}\hat{u}, \frac{1}{2}\bar{u}' + \frac{1}{2}\hat{u}'\right) \right\} dx = 0$$

Now, by convexity of L, the INTEGRAND in $(*)$ is always ≥ 0. Hence, by continuity, we conclude that

$$L\left(x, \frac{1}{2}\bar{u} + \frac{1}{2}\hat{u}, \frac{1}{2}\bar{u}' + \frac{1}{2}\hat{u}'\right) = \frac{1}{2} L(x, \bar{u}, \bar{u}') + \frac{1}{2} L(x, \hat{u}, \hat{u}'), \quad \forall x \in [a,b]$$

Since L is STRICTLY CONVEX, the above is possible iff $\bar{u}(x) = \hat{u}(x)$ and $\bar{u}'(x) = \hat{u}'(x)$, $\forall x \in [a,b]$.

Thus $\bar{u} \equiv \hat{u}$ and the minimizer is unique. $\square$

EXAMPLE Let $L : \mathbb{R} \to \mathbb{R}$, $L = L(p)$. Assume $L \in C^2(\mathbb{R})$. Define

$$X := \{ u \in C^1[a,b] \mid u(a) = \alpha, \ u(b) = \beta \}$$

$$V := \{ v \in C^1[a,b] \mid v(a) = 0, \ v(b) = 0 \}$$

Consider $F: X \to \mathbb{R}$ defined by

$$F(u) := \int_a^b L(\dot{u})\,dx$$

We can then write ELE in DIFFERENTIAL FORM:

$$\begin{cases} \dfrac{d}{dx}\Big[L_p(x, u(x), \dot{u}(x))\Big] = L_s(x, u(x), \dot{u}(x)) \quad , \quad \forall x \in (a,b) \\ \\ u(a) = \alpha \,, \quad u(b) = \beta \end{cases}$$

which in this case reads

$$\text{(ELE)} \quad \begin{cases} \dfrac{d}{dx}\Big[L'(\dot{u}(x))\Big] = 0 \quad , \quad \forall x \in (a,b) \\ \\ u(a) = \alpha \,, \quad u(b) = \beta . \end{cases}$$

Now, the above ODE implies that

$$\boxed{L'(\dot{u}) = \text{CONSTANT}}$$

Therefore the straight line

$$\bar{u}(x) := \frac{\beta - \alpha}{b - a}(x - a) + \alpha$$

is ALWAYS a solution to (ELE).

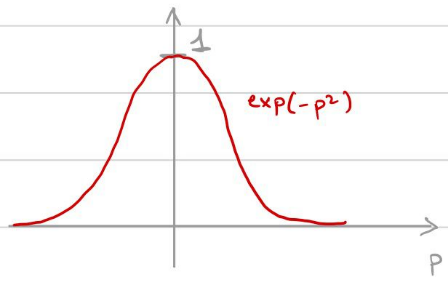
QUESTION When does $\bar{u}$ also solve

$$\text{(P)} \qquad F(\bar{u}) = \min \{ F(u) \mid u \in X \} \qquad ?$$

CASE 1 Assume L CONVEX. As $\bar{u}$ solves (ELE), in particular it solves ELE in INTEGRAL FORM. Then by THEOREM 5.4 we have that $\bar{u}$ solves (P).

CASE 2 If we do not assume convexity, then in general $\bar{u}$ DOES NOT solve (P). For example let

$$L(p) := \exp(-p^2)$$



Let us consider the case with zero Dirichlet conditions, i.e.,

$$X = V = \{ u \in C^1[0,1] \mid u(0) = u(1) = 0 \}.$$

Note that in this setting our straight line is $\bar{u} \equiv 0$. Then $\bar{u}$ solves (ELE), but is it solution to (P) ?

Clearly L is not convex, so THEOREM 5.4 cannot be applied.

FACT The minimization problem (P) has NO SOLUTION and

$$m := \inf \{ F(u) \mid u \in X \} = 0$$

(This will be left as an exercise)

Therefore $\bar{u} \equiv 0$ solves (ELE) but DOES NOT solve (P).

② TRIVIAL LEMMA

Given $\bar{u}$ solution to ELE, we want to know if $\bar{u}$ is also a minimizer. A possible way to answer this question is given by the following Lemma.

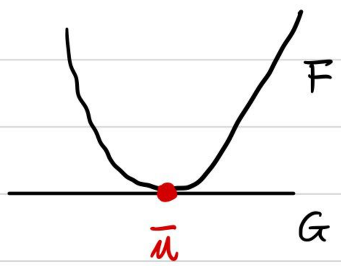
LEMMA 5.5 (LEMMA TRIVIAL)

Let X be a set and $F, G : X \to \mathbb{R}$ functionals. Assume that

(i) $F(u) \geq G(u)$, $\forall u \in X$

(ii) $\bar{u} \in X$ is a minimizer for G on X

(iii) $F(\bar{u}) = G(\bar{u})$.



Then $\bar{u}$ is a minimizer for F. If in addition $\bar{u}$ is the unique minimizer of G, then $\bar{u}$ is the unique minimizer of F.

Proof Let $u \in X$ be arbitrary. Then

$$F(u) \overset{(i)}{\geq} G(u) \overset{(ii)}{\geq} G(\bar{u}) \overset{(iii)}{=} F(\bar{u}),$$

showing that $\bar{u}$ minimizes F.

Assume now that $\bar{u}$ is the unique minimizer of G. Then, for the first part of the statement, we know that $\bar{u}$ also minimizes F. Suppose that $\bar{w} \in X$ is another minimizer for F. Then

$$G(\bar{w}) \underset{(i)}{\leq} F(\bar{w}) \underset{\substack{\text{minimality of } \bar{u} \\ \text{and } \bar{w} \text{ for } F}}{=} F(\bar{u}) \overset{(iii)}{=} G(\bar{u})$$

Thus $G(\bar{w}) = G(\bar{u})$, being $\bar{u}$ minimizer for G. $\Rightarrow \bar{u} = \bar{w}$ as the minimizer of G is unique. $\square$

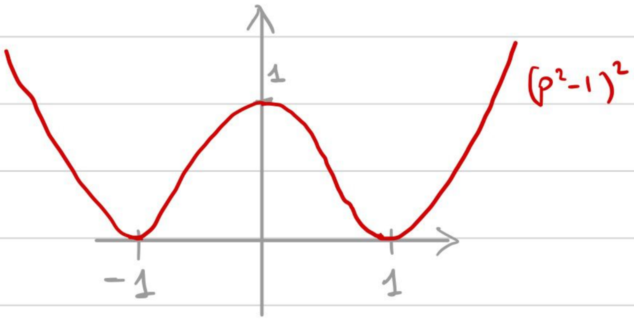
COMMENT: The above lemma requires to find a functional G satisfying (i), (ii), (iii). This is not always obvious. However in the future we will see a systematic way to construct G from F.

EXAMPLE 5.6 $X = \{ u \in C^1[0,1] \mid u(0) = 1, \ u(1) = 3 \}$

$F: X \to \mathbb{R}$ defined by

$$F(u) := \int_0^1 (\dot{u}^2 - 1)^2 \, dx$$

Note that the Lagrangian is $L = L(p) = (p^2 - 1)^2$, which is not convex. Such Lagrangian is very typical and is named DOUBLE WELL.



DOUBLE WELL

The ELE for the minimum problem associated to F is

$$\begin{cases} \frac{d}{dx} L_p(\dot{u}) = 0, & \forall x \in (0,1) \\ u(0) = 1, \ u(1) = 3 \end{cases}$$

From $L_p(\dot{u})' = 0$ we deduce $L_p(\dot{u}) = \text{CONSTANT}$. Therefore the line

$$\bar{u}(x) := 2x + 1$$

Satisfies the BOUNDARY CONDITIONS and ELE.

NOTE If L was CONVEX we could have concluded that $\bar{u}$ minimizes F, by THEOREM 5.4. However L is not convex, so we need to proceed differently.

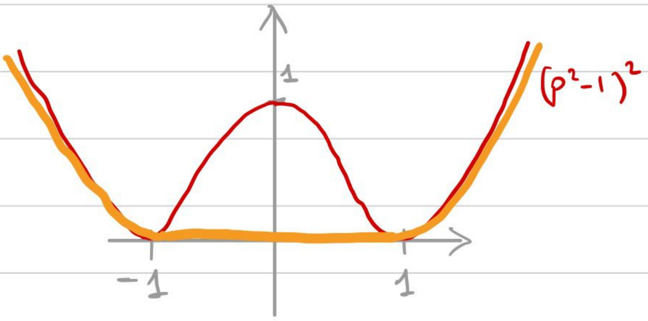
CLAIM $\bar{u}$ is the unique minimizer of F in X.

Proof We make use of the TRIVIAL LEMMA. We need to find G satisfying the assumptions.

IDEA: Find a Lagrangian $\hat{L}$ such that $\hat{L} \le L$ and that the functional

$$G(u) := \int_0^1 \hat{L}(\dot{u})\, dx \quad , \quad u \in X$$

is likely to admit unique minizer. Ideally we also want $\hat{L}$ to be CONVEX, so we can apply THEOREM 5.4 to G.



A good idea is then to convexify L, by setting

$$\hat{L}(p) := \begin{cases} L(p) & \text{if } |p| \ge 1 \\ 0 & \text{if } |p| \le 1 \end{cases}$$

Notice that $\hat{L}$ is convex. We now verify (i), (ii), (iii) from LEMMA 5.5:

(i) $F(u) \le G(u)$, $\forall u \in X$: True because $\hat{L} \le L$ pointwise.

(ii) $\bar{u}$ minimizes G: True because $\hat{L}$ depends only on p. Therefore the line $\bar{u}$ is solution of ELE for G:

$$\begin{cases} \dfrac{d}{dx} \hat{L}_p(\bar{u}') = 0 \\ \bar{u}(0) = 1, \quad \bar{u}(1) = 2 \end{cases}$$

Therefore $\bar{u}$ minimizes G by THEOREM 5.4, as $\hat{L}$ is CONVEX.

(iii) $F(\bar{u}) = G(\bar{u})$: True because $\bar{u}' \equiv 2$, and $\hat{L}(2) = L(2)$ by definition.

Therefore $\bar{u}$ minimizes F by LEMMA TRIVIAL 5.5.

Also note that $\hat{L}$ is STRICTLY CONVEX in a neighborhood of $p=2$ (that is, in a neighborhood of $\bar{u}' \equiv 2$). Thus (by a slightly more general version of THEOREM 5.4 we conclude that $\bar{u}$ is the unique minimizer of G.
By Lemma 5.5 we then have that $\bar{u}$ is the unique minimizer of F. □

EXAMPLE 5.7 (VARIATION ON EXAMPLE 5.6)

Let us consider the same Lagrangian $L(p) = (p^2-1)^2$ as in EXAMPLE 5.6

However this time we look for a minimum of F over the set

$$X = \{ u \in C^1[a,b] \mid u(0) = 0 , u(1) = 0 \}$$

Note : The only difference is we have changed the DIRICHLET BC

Let's try to show that the line passing through $(0,0)$ and $(1,0)$, i.e.,

$$\bar{u}(x) \equiv 0$$

(which solves ELE associated to F) is a minimizer for F.
We immediately see that the above strategy fails, because by definition

$$\hat{L}(0) = 0 , \quad \text{while} \quad L(0) = 1$$

Thus (iii) does not hold and we cannot apply LEMMA 5.5 to F, G and $\bar{u}$.

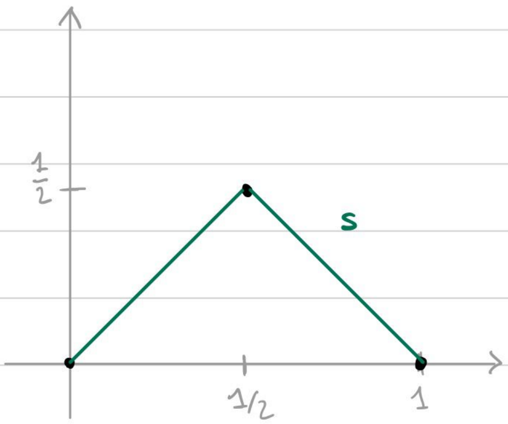
What is going on ?

Since the constraints are $u(0)=0$, $u(1)=0$, then $\bar{u}$ is NOT a minimizer for F. More in general:

CLAIM F admits no minimizer in X. Moreover

$$m := \inf \{ F(u) \mid u \in X \} = 0$$

Proof The idea is that, since $L(1) = L(-1) = 0$, we can construct a function $\tilde{s} \in X$ s.t. $|\tilde{s}'| \approx 1$ and so $F(\tilde{s}) \approx 0$. This is possible because the points $(0,0)$, $(1,0)$ are sufficiently close. To construct $\tilde{s}$, define $s : [0,1] \to \mathbb{R}$ by



$$s(x) := \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2} \\ -x+1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Notice that $s(0) = 0$, $s(1) = 0$ and $s' \in \{-1, 1\}$. Thus $F(s) = 0$.

The only problem is that s is not C^1. However, we can "ROUND the CORNER" at $x = 1/2$ by paying a small amount of energy (see WORKSHEET 3)

Thus we can define $\tilde{s} : [0,1] \to \mathbb{R}$ s.t.

$$\tilde{s} \in C^1[0,1], \quad \tilde{s}(0) = \tilde{s}(1) = 0, \quad \tilde{s}'(x) = \pm 1 \text{ for } x \in [0,1] \setminus I, \quad F(\tilde{s}) = \varepsilon$$

with $I = (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ for some $\delta > 0$ and $\varepsilon > 0$ arbitrary. Then $\tilde{s} \in X$, so that $m \leq F(\tilde{s}) = \varepsilon$. As ε is arbitrary, we conclude $m = 0$ (as $F \geq 0$).

Finally, to see that the infimum is not attained, if it existed $\bar{u} \in X$ s.t. $F(\bar{u}) = 0$, then in particular

$$F(\bar{u}) = \int_0^1 ((\bar{u}')^2 - 1)^2 \, dx = 0 \implies \bar{u}' \in \{-1, 1\} \text{ for all } x \in [0,1].$$

However, as $\bar{u}'$ is continuous, we can only have $\bar{u}' \equiv 1$, or $\bar{u}' \equiv -1$, which are not possible since we must have $\bar{u}(0) = \bar{u}(1) = 0$ by the DIRICHLET BC. Thus F admits no minimizer. $\square$

NOTE In general if we define $X := \{ u \in C^1[0,1] \mid u(0) = \alpha, \, u(1) = \beta \}$ and

$$F(u) := \int_0^1 (\dot{u}^2 - 1)^2 \, dx, \qquad u \in X.$$

then

- If $|\beta - \alpha| > 1$, then the unique minimizer of F is the straight line

$$\bar{u}(x) = (\beta - \alpha) x + \alpha$$

  which can be shown as in EXAMPLE 5.6.

- If $|\beta - \alpha| \leq 1$ then F admits no minimizers and the infimum is 0. This can be shown by adapting the arguments of EXAMPLE 5.7.

SUMMARY OF INDIRECT METHOD

Given a minimization problem, the strategy is as follows:

1. Finding necessary conditions for minimality: ELE + BC
2. Solve ELE + BC (This is possible in very few cases: linear differential equations and not much more)

3. Prove that STATIONARY POINTS found in 2 are minimizers:

   → Using CONVEXITY
   → Using TRIVIAL LEMMA

7. SOBOLEV SPACES

REFERENCE:
GIOVANNI LEONI – "A FIRST COURSE IN SOBOLEV SPACES"
AMERICAN MATHEMATICAL SOCIETY, 2017

So far our minimization problems were mostly set in C^1. Often a solution did not exist in C^1, however in many examples we saw that we could find an infimizing sequence converging to something piecewise C^1, i.e.,

$$C^1_{pw}[a,b] := \{ u \in C[a,b] \mid \exists \{x_0, \dots, x_N\} \subseteq [a,b] \text{ with } a = x_0 < x_1 < \dots < x_N = b, \; u \in C^1[x_i, x_{i+1}], \; i = 0, \dots, N-1 \}$$

However these functional spaces are not very convenient to work with, due to their lack of completeness WRT weaker norms (e.g. the L^p convergence). The default functional spaces for setting variational problems are (nowadays and in the past 60-70 years) SOBOLEV SPACES.

In order to define Sobolev spaces, we rely on previous knowledge about L^p spaces (LEBESGUE SPACES). A self-contained summary of definitions and properties can be found in SECTION 6 of these notes. (L^p SPACES REVISION)

Here we just recall the definition of L^p spaces, to establish some notation.

L^p SPACES Let $(X, \mathcal{A}, \mu)$ be a measurable space, where X set, $\mathcal{A}$ is σ-algebra over X and $\mu : \mathcal{A} \to [0, +\infty]$ is a measure. For $1 \leq p < +\infty$ and $p = +\infty$ we set, respectively:

$$L^p(X, \mu) := \left\{ u : X \to \mathbb{R} \mid u \text{ measurable}, \; \int_X |u|^p \, d\mu < +\infty \right\}$$

$$L^\infty(X, \mu) := \{ u : X \to \mathbb{R} \mid u \text{ measurable}, \; \exists C > 0 \text{ s.t. } |u(x)| \leq C \;\; \mu\text{-a.e. in } X \}$$

When we say μ-a.e. we mean that a certain property holds in $X \setminus E$, where $\mu(E) = 0$.

WARNING The elements of $L^p(X,\mu)$ and $L^\infty(X,\mu)$ are <u>NOT</u> functions, but classes of equivalence of functions, where the equivalence is

$$u \sim v \iff u(x) = v(x) \quad \text{for } \mu\text{-a.e. } x \in X$$

This is not an issue, since in this case $\int_X u \, d\mu = \int_X v \, d\mu$.
Therefore $L^p(X,\mu)$ and $L^\infty(X,\mu)$ have to be understood as QUOTIENT SPACES WRT $\sim$

RECALL $L^p(X,\mu)$, $L^\infty(X,\mu)$ are Banach spaces with the norms

$$\|u\|_p := \left(\int_X |u|^p \, d\mu \right)^{1/p}, \quad u \in L^p(X,\mu), \quad 1 \leq p < +\infty,$$

$$\|u\|_\infty := \inf \{ C : |u(x)| \leq C \ \mu\text{-a.e. in } X \}, \quad u \in L^\infty(X,\mu)$$

Moreover $L^2(X,\mu)$ is a Hilbert space with inner product

$$\langle u, v \rangle := \int_X u v \, d\mu, \quad u, v \in L^2(X,\mu)$$

NOTE In the following the definition of L^p will be employed in this setting:

- X will always be an OPEN SET OF $\mathbb{R}^d$
- $\mathcal{A}$ is the d-dimensional LEBESGUE σ-Algebra
- $\mu = dx = \mathcal{L}^d$ the d-dimensional LEBESGUE MEASURE

Thus we will always write $L^p(X)$ in place of $L^p(X,\mu)$, as there is no ambiguity.

We need to introduce versions of the FLCV and DBR Lemma for L^p functions.
For that we need tools to smoothen functions, i.e., convolutions.

CONVOLUTIONS

DEFINITION 7.1 Let $u, v : \mathbb{R} \to \mathbb{R}$. The **CONVOLUTION** between u and v, is defined as

$$\boxed{(u * v)(x) := \int_{\mathbb{R}} u(x-y)\, v(y)\, dy}$$

for all $x \in \mathbb{R}$ s.t. the RHS is FINITE.

REMARK It is immediate to check that, whenever the convolution is finite,

$$u * v = v * u \qquad \text{and} \qquad u * (v * w) = (u * v) * w$$

for $u, v, w : \mathbb{R} \to \mathbb{R}$.

The following Theorem gives a sufficient condition for $u * v$ to be well-defined.

THEOREM 7.2 (YOUNG)

Let $u \in L^1(\mathbb{R})$, $v \in L^p(\mathbb{R})$ for some $1 \le p \le +\infty$. Then for a.e. $x \in \mathbb{R}$ the map $y \mapsto u(x-y)\, v(y)$ is integrable, so that $u * v$ is finite. Moreover $u * v \in L^p(\mathbb{R})$, with

$$(*) \qquad \| u * v \|_p \le \| u \|_1 \, \| v \|_p .$$

Proof • $p = +\infty$: this is immediate, since for a.e. $x \in \mathbb{R}$

$$|(u * v)(x)| \le \int_{\mathbb{R}} |u(x-y)|\, |v(y)|\, dy \le \|v\|_\infty \int_{\mathbb{R}} |u(x-y)|\, dy = \|v\|_\infty \, \|u\|_1 .$$

Taking the essential supremum in the above inequality we obtain $(*)$

• $p=1$: Set $\psi(x,y) := u(x-y)v(y)$. For a.e. $y \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} |\psi(x,y)|\, dx = |v(y)| \int_{\mathbb{R}} |u(x-y)|\, dx = |v(y)|\, \|u\|_1 < +\infty$$

Integrating WRT x we get

$$(**) \quad \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dx \right\} dy = \|v\|_1 \|u\|_1 < +\infty$$

Then ψ satisfies the assumptions of TONELLI'S THEOREM (THEOREM 6.9) and we infer $\psi \in L^1(\mathbb{R}\times\mathbb{R})$ (where $\mathbb{R}\times\mathbb{R}$ is equipped with the 2-dimensional Lebesgue measure). We can then apply FUBINI'S THEOREM (THEOREM 6.10) to get that

$$\int_{\mathbb{R}} |\psi(x,y)|\, dy < +\infty \quad \text{for a.e. } x \in \mathbb{R}.$$

and also

$$(***) \quad \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dy \right\} dx \overset{\text{FUBINI}}{=} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dx \right\} dy \overset{(**)}{=} \|v\|_1 \|u\|_1$$

Therefore

$$\int_{\mathbb{R}} \left| (u * v)(x) \right| dx \overset{\text{def. of convolution}}{=} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} u(x-y)v(y)\, dy \right| dx \leq \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |u(x-y)v(y)|\, dy \right\} dx$$

$$\text{def of } \psi \rightarrow = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\psi(x,y)|\, dy \right\} dx \overset{(***)}{=} \|u\|_1 \|v\|_1 ,$$

which is exactly $(*)$.

• $1 < p < +\infty$: The functions $|u|, |v|^p \in L^1(\mathbb{R})$. Thus, from the case $p=1$, we know that $y \mapsto |u(x-y)|\, |v(y)|^p$ belongs to $L^1(\mathbb{R})$ for a.e. $x \in \mathbb{R}$.
In particular

$$|u(x-\cdot)|^{1/p}\, |v(\cdot)| \in L^p(\mathbb{R}) \quad \text{for a.e. } x \in \mathbb{R}.$$

Moreover, as $u \in L^1(\mathbb{R})$, we also have

$$|u(x-\cdot)|^{1/p'} \in L^{p'}(\mathbb{R}) \quad \text{for a.e. } x \in \mathbb{R}$$

where we chose p' as the HÖLDER CONJUGATE, i.e.

$$p' := \frac{p}{p-1}, \quad \text{so that} \quad \frac{1}{p} + \frac{1}{p'} = 1$$

From HÖLDER INEQUALITY (THEOREM 6.11) we get, for a.e. $x \in \mathbb{R}$,

$$|(u * v)(x)| \leq \int_{\mathbb{R}} |u(x-y)|\, |v(y)|\, dy$$

$$= \int_{\mathbb{R}} \underbrace{|u(x-y)|^{1/p'}}_{\in L^{p'}} \underbrace{|u(x-y)|^{1/p}\, |v(y)|}_{\in L^p} dy$$

(HÖLDER) $$\leq \left(\int_{\mathbb{R}} |u(x-y)|\, dy \right)^{1/p'} \left(\int_{\mathbb{R}} |u(x-y)|\, |v(y)|^p dy \right)^{1/p}$$

$$= \|u\|_1^{1/p'} \cdot \left[(|u| * |v|^p)(x) \right]^{1/p}$$

Taking the p-power of the above we get

$$(**)\quad |(u*v)(x)|^p \leq \|u\|_1^{p/p'} \, (|u| * |v|^p)(x) \quad \text{for a.e. } x \in \mathbb{R}$$

Now, as $|u|, |v|^p \in L^1(\mathbb{R})$, we can apply $(*)$ for the case $p=1$ to get:

$$(***)\quad \| \, |u| * |v|^p \, \|_1 \leq \|u\|_1 \, \| \, |v|^p \, \|_1 = \|u\|_1 \, \|v\|_p^p$$

By integrating $(**)$:

$$\int_{\mathbb{R}} |(u*v)(x)|^p \, dx \overset{(**)}{\leq} \|u\|_1^{p/p'} \int_{\mathbb{R}} |(|u| * |v|^p)(x)| \, dx$$

and so

$$= \|u\|_1^{p/p'} \, \| \, |u| * |v|^p \, \|_1$$

$$\overset{(***)}{\leq} \|u\|_1^{p/p'} \, \|u\|_1 \, \|v\|_p^p$$

As $\frac{p}{p'} + 1 = p$ $\longrightarrow$ $= \|u\|_1^p \, \|v\|_p^p$.

Taking the $\frac{1}{p}$ - power of the above inequality yields $(*)$. $\square$

We now need the notion of SUPPORT for L^p functions. Indeed, as elements of L^p are actually equivalence classes, and thus defined a.e., the definition of support we used for continuous functions makes no sense:

EXAMPLE $u := \chi_{\mathbb{Q}}$. As the Lebesgue measure of $\mathbb{Q}$ is zero, u belongs to the same equivalence class of $v \equiv 0$. Using the classical definition of support we get

$$\operatorname{supp} u = \overline{\{x \in \mathbb{R} \mid u(x) \neq 0\}} = \overline{\mathbb{Q}} = \mathbb{R}, \quad \text{while} \quad \operatorname{supp} v = \emptyset .$$