6. L^p SPACES REVISION

REFERENCE W. RUDIN – "REAL AND COMPLEX ANALYSIS"
Mc GRAW – HILL, 2001

MEASURE THEORY

σ- Algebra Let Ω be a SET. Denote by $P(\Omega)$ the set of all subsets of Ω. A collection $\mathcal{A} \subseteq P(\Omega)$ is called a σ- ALGEBRA if

(1) $\phi \in \mathcal{A}$

(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$, where $A^c := \Omega \setminus A$

(3) If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then $\bigcup_{n=1}^{+\infty} A_n \in \mathcal{A}$

The sets in $\mathcal{A}$ are called MEASURABLE.
The pair $(\Omega, \mathcal{A})$ is called MEASURE SPACE

NOTATION If $\mathcal{G} \subseteq P(\Omega)$ is a collection of sets, we denote by $\sigma(\mathcal{G})$ the smallest σ- algebra on Ω containing $\mathcal{G}$, that is,

$$\sigma(\mathcal{G}) := \bigcap \{\mathcal{A} \subseteq P(\Omega) \mid \mathcal{A} \text{ is } \sigma\text{-algebra}, \mathcal{G} \subseteq \mathcal{A}\}$$

BOREL SETS If τ is a topology over Ω, we call $\sigma(\tau)$ the BOREL σ-algebra. The elements of $\sigma(\tau)$ are called BOREL SETS

MEASURES A set function $\mu : \mathcal{A} \to [0, +\infty]$ is called a MEASURE if

(1) $\mu(\phi) = 0$

COUNTABLY ADDITIVE → (2) $\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \sum_{n=1}^{+\infty} \mu(A_n)$ whenever $\{A_n\} \subseteq \mathcal{A}$ and they are pairwise disjoint, i.e., $A_i \cap A_j = \phi$ if $i \neq j$

The triple $(\Omega, \mathcal{A}, \mu)$ is called MEASURABLE SPACE

TERMINOLOGY Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space.

- μ is called **COMPLETE** if for all $E \in \mathcal{A}$ s.t. $\mu(E)=0$, then every $F \subseteq E$ satisfies $F \in \mathcal{A}$.

- μ is **σ-FINITE** if $\exists \{\Omega_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ s.t.

$$\Omega = \bigcup_{n=1}^{+\infty} \Omega_n \quad \text{and} \quad \mu(\Omega_n) < +\infty, \ \forall n \in \mathbb{N}.$$

- μ is **FINITE** if $\mu(\Omega) < +\infty$

- The sets $E \in \mathcal{A}$ s.t. $\mu(E)=0$ are called **NULL SETS**

- We say that a property holds **μ-ALMOST EVERYWHERE** in Ω (abbreviated in μ-a.e.) if $\exists E \in \mathcal{A}$ s.t. $\mu(E)=0$ and the property holds for all $x \in \Omega \setminus E$.

OUTER MEASURES Ω set. A set map $\mu^*: P(\Omega) \to [0,+\infty]$ is called **OUTER MEASURE** if

(a) $\mu^*(\emptyset)=0$

Monotonic → (b) $\mu^*(E) \leq \mu^*(F)$ for all $E \subseteq F \subseteq \Omega$

Sub-additive → (c) $\mu\left(\bigcup_{n=1}^{+\infty} E_n\right) \leq \sum_{n=1}^{+\infty} \mu^*(E_n)$, for all $\{E_n\}_{n\in\mathbb{N}} \subseteq \Omega$

To construct an outer measure we usually start with a family $\mathcal{G} \subseteq P(\Omega)$ of elementary sets (e.g. cubes in $\mathbb{R}^d$), for which we have a desired notion of measure $\rho: \mathcal{G} \to [0,+\infty]$.

PROPOSITION 6.1 Let $\Omega \neq \emptyset$, $\mathcal{G} \subseteq \mathcal{P}(\Omega)$, $\rho : \mathcal{G} \to [0, +\infty]$. Assume that

- $\emptyset \in \mathcal{G}$ and $\rho(\emptyset) = 0$,
- $\exists \{\Omega_n\}_{n \in \mathbb{N}} \subseteq \mathcal{G}$ s.t. $\Omega = \bigcup_{n=1}^{+\infty} \Omega_n$

Define $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ by

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{+\infty} \rho(E_n) \;\middle|\; \{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{G},\ E \subseteq \bigcup_{n=1}^{+\infty} E_n \right\}$$

Then μ^* is an OUTER MEASURE.

The problem with outer measures is that they are not additive on disjoint sets. To solve this problem, we restrict μ^* on a smaller collection of sets $\mathcal{A}^* \subseteq \mathcal{P}(\Omega)$:

μ^*-MEASURABLE SETS Given $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ outer measure, we say that $E \subseteq \Omega$ is **μ^*-MEASURABLE** if

$$\mu^*(E) = \mu^*(E \cap F) + \mu^*(E \cap F^c), \quad \forall F \subseteq \Omega$$

THEOREM 6.2 **(CARATHÉODORY)**

Let $\Omega \neq \emptyset$ and let $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ be an outer measure. Define

$$\mathcal{A}^* := \{ E \subseteq \Omega \mid E \text{ is } \mu^*\text{-measurable} \}.$$

Then $\mathcal{A}^*$ is a σ-algebra and $\mu^* : \mathcal{A}^* \to [0, +\infty]$ is a COMPLETE MEASURE.

THE LEBESGUE MEASURE On $\mathbb{R}^d$ we can construct a particular measure called the LEBESGUE MEASURE.

For $x \in \mathbb{R}^d$, $r > 0$, define $Q(x,r) := x + \left(-\frac{r}{2}, \frac{r}{2}\right)^d$, the CUBE of side length r centered at x. Introduce the collection of cubes $\mathcal{G} \subseteq \mathcal{P}(\mathbb{R}^d)$ as

$$\mathcal{G} := \{ Q(x,r) \mid x \in \mathbb{R}^d, r > 0 \} \cup \{\emptyset\}$$

and $\rho : \mathcal{G} \to [0, +\infty)$ s.t. $\rho(\emptyset) := 0$ and $\rho(Q(x,r)) := r^d$. We can then define $\mathcal{L}_0^d : \mathcal{P}(\mathbb{R}^d) \to [0, +\infty]$ as

$$\mathcal{L}_0^d(E) := \inf \left\{ \sum_{i=1}^{+\infty} r_i^d \;\middle|\; E \subseteq \bigcup_{i=1}^{+\infty} Q(x_i, r_i) \right\}$$

$$= \inf \left\{ \sum_{i=1}^{+\infty} \rho(E_i) \;\middle|\; \{E_i\} \subseteq \mathcal{G},\ E \subseteq \bigcup_{i=1}^{+\infty} E_i \right\}$$

i.e., cover E with cubes and sum up the volumes (counting overlapping). Then take the smallest outcome.

By PROPOSITION 6.1 we have that $\mathcal{L}_0^d$ is an outer measure, called the **LEBESGUE OUTER MEASURE**. It can be shown that

- $\mathcal{L}_0^d(Q(x,r)) = r^d$
- $\mathcal{L}_0^d$ is TRANSLATION INVARIANT:

$$\mathcal{L}_0^d(x+E) = \mathcal{L}_0^d(E), \quad \forall x \in \mathbb{R}^d, E \subseteq \mathbb{R}^d$$

Define

$$\mathcal{L}^* := \{ E \subseteq \mathbb{R}^d \mid E \text{ is } \mathcal{L}_0^d\text{-measurable} \}$$

Then by THEOREM 6.2 we have that:

1. $\mathcal{L}^*$ is a σ-algebra, called the **σ-ALGEBRA of LEBESGUE MEASURABLE SETS**

2. $\mathcal{L}_o^d$ restricted to $\mathcal{L}^*$ is a COMPLETE MEASURE. We denote it by $\mathcal{L}^d$ and call it the **d-DIMENSIONAL LEBESGUE MEASURE**

Notice that $\mathcal{L}^d$ is <u>not</u> FINITE $(\mathcal{L}(\mathbb{R}^d) = +\infty)$ but it is σ-FINITE, since

$$\mathbb{R}^d = \bigcup_{n=1}^{+\infty} Q(0,n) \quad \text{and} \quad \mathcal{L}^d(Q(0,n)) = n^d < +\infty .$$

Moreover, if we denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra of $\mathbb{R}^d$ WRT the Euclidean topology, we have

$$\boxed{\mathcal{B}(\mathbb{R}^d) \subseteq \mathcal{L}^*}$$

i.e., all Borel sets of $\mathbb{R}^d$ are Lebesgue measurable.

<u>WARNING</u> The inclusion $\mathcal{B}(\mathbb{R}^d) \subsetneq \mathcal{L}^*$ is STRICT : $\exists$ sets in $\mathcal{L}^*$ which is not Borel measurable. Thus $\mathcal{L}^d$ restricted to $\mathcal{B}(\mathbb{R}^d)$ is <u>not</u> COMPLETE.

<u>WARNING</u> There exist sets $E \subseteq \mathbb{R}^d$ which are <u>NOT</u> Lebesgue measurable.

INTEGRABILITY On a measurable space $(\Omega, \mathcal{A}, \mu)$ we can define the notion of integrability.

MEASURABLE FUNCTIONS Let X, Y be non-empty sets, $\mathcal{A}$ and $\mathcal{B}$ be σ-algebras on X and Y respectively. A function $u : X \to Y$ is **MEASURABLE** if

$$u^{-1}(E) \in \mathcal{A} \quad \text{for all} \quad E \in \mathcal{B}.$$

If X, Y are topological spaces and $\mathcal{A}, \mathcal{B}$ are Borel σ-algebras then measurable functions are called **BOREL FUNCTIONS**.

REMARK 6.3 ① If $(X, \mathcal{A})$ is a measurable space and $u: X \to \mathbb{R}$ with $\mathbb{R}$ equipped with the Borel σ-algebra, then

$$u \text{ is measurable} \iff u^{-1}((a, +\infty)) \in \mathcal{A} \text{ for every } a \in \mathbb{R}.$$

If instead $u : X \to [-\infty, +\infty]$ (always with Borel σ-algebra) then

$$u \text{ is measurable} \iff u^{-1}((a, +\infty]) \in \mathcal{A} \text{ for every } a \in \mathbb{R}.$$

② If X, Y are topological spaces equipped with Borel σ-algebras then

$$u : X \to Y \text{ CONTINUOUS} \Rightarrow u \text{ Borel}$$

③ The composition of measurable functions is measurable. In particular if $(X, \mathcal{A})$ is a measure space and $u : X \to \mathbb{R}$ is measurable, then u^p, $|u|$, cu and

$$u^+ := \begin{cases} u & \text{if } u(x) \geq 0 \\ 0 & \text{if } u(x) < 0 \end{cases}, \quad u^- := \begin{cases} -u & \text{if } u(x) \leq 0 \\ 0 & \text{if } u(x) > 0 \end{cases}$$

are all measurable, for $p \geq 1$, $c \in \mathbb{R}$.

④ Moreover if $v : X \to \mathbb{R}$ is measurable then $u+v$, uv, $\min\{u,v\}$, $\max\{u,v\}$ are measurable.

⑤ Let $(X, \mathcal{A})$ be a measurable space and $u_n : X \to [-\infty, +\infty]$ be measurable. Then the functions

$$\sup_{n\in\mathbb{N}} u_n \quad , \quad \inf_{n\in\mathbb{N}} u_n \quad , \quad \liminf_{n\to+\infty} u_n \quad , \quad \limsup_{n\to+\infty} u_n$$

are measurable.

⑥ Let $(X, \mathcal{A}, \mu)$ be a measurable space. Assume that μ is COMPLETE. If $u_n : X \to [-\infty, +\infty]$ are measurable and

$$u(x) := \lim_{n\to+\infty} u_n(x) \quad \text{exists for } \mu\text{-a.e. } x \in X$$

then u is measurable.

⑦ Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measurable spaces and $\mu : \mathcal{A} \to [0, +\infty]$ be a measure. Suppose $u : X \to Y$ is measurable. If $v : X \to Y$ is s.t.

$$u(x) = v(x) \quad \text{for } \mu\text{-a.e. } x \in X$$

then v is also measurable.

We are now ready to introduce integrals. For a measurable space $(X, \mathcal{A})$ and $E \subseteq X$ we define the CHARACTERISTIC FUNCTION of E as

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Note that χ_E is measurable if $E \in \mathcal{A}$.

SIMPLE FUNCTIONS $(X, \mathcal{A})$ measurable space. A SIMPLE FUNCTION is a measurable map $s: X \to \mathbb{R}$ such that $s(X)$ is finite, i.e., there exist disjoint sets $E_1, .., E_N \in \mathcal{A}$, $N \in \mathbb{N}$ and $c_1, .., c_N \in \mathbb{R}$ distinct, s.t.

$$(*) \qquad s(x) = \sum_{i=1}^{N} c_i \chi_{E_i}(x) \quad , \quad \forall x \in X.$$

THEOREM 6.4 $(X, \mathcal{A})$ measure space, $u: X \to [0, +\infty]$ measurable. Then there exists a sequence $\{s_n\}$ of SIMPLE FUNCTIONS s.t. $0 \le s_1 \le s_2 \le \dots$ and $s_n(x) \to u(x)$ for all $x \in X$.

LEBESGUE INTEGRAL Let $(X, \mathcal{A}, \mu)$ be a measurable space. The LEBESGUE INTEGRAL is defined in 3 steps:

① Let $s \geq 0$ a step function of the form $(*)$. We define the LEBESGUE INTEGRAL of s on a set $E \in \mathcal{A}$ by

$$\boxed{\int_E s(x)\, d\mu(x) := \sum_{i=1}^{N} c_i\, \mu(E \cap E_i)}$$

where if $c_i = 0$ and $\mu(E \cap E_i) = +\infty$ we adopt the standard convention

$$c_i\, \mu(E \cap E_i) := 0.$$

② Let $u: X \to [0, +\infty]$ be a measurable function (note that $u \geq 0$). The LEBESGUE INTEGRAL of u over a set $E \in \mathcal{A}$ is defined as

$$\boxed{\int_E u(x)\, d\mu(x) := \sup \left\{ \int_E s\, d\mu \;\middle|\; s \text{ simple},\ 0 \le s \le u \right\}}$$

(This is well posed thanks to THEOREM 6.4)

③ Let $u : X \to [-\infty, +\infty]$ be measurable. Note that $u = u^+ - u^-$ with $u^+, u^- \geq 0$. The LEBESGUE INTEGRAL of u over a set $E \in \mathcal{A}$ is defined

$$\boxed{\int_E u(x)\, d\mu := \int_E u^+\, d\mu - \int_E u^-\, d\mu}$$

If $\int_E u^+ d\mu$ and $\int_E u^- d\mu$ are FINITE then u is said to be LEBESGUE INTEGRABLE WRT μ.

REMARK Let $(X, \mathcal{A}, \mu)$ be a measurable space. Let $u, v : X \to [-\infty, +\infty]$ be measurable.

① If $0 \leq u \leq v$ then $\int_E u\, d\mu \leq \int_E v\, d\mu$, $\forall E \in \mathcal{A}$

② If $c \in [0, +\infty]$, then $\int_E cu\, d\mu = c \int_E u\, d\mu$, $\forall E \in \mathcal{A}$ $(0 \cdot (\pm\infty) := 0)$

③ Let $E \in \mathcal{A}$ and $u \geq 0$. Then $\int_E u\, d\mu = 0$ iff $u(x) = 0$ for μ-a.e. $x \in E$.

④ If $E \in \mathcal{A}$ and $\mu(E) = 0$ then $\int_E u\, d\mu = 0$

⑤ If $E \in \mathcal{A}$ then $\int_E u\, d\mu = \int_X \chi_E u\, d\mu$

⑥ u is LEBESGUE INTEGRABLE iff $\int_E |u|\, d\mu < +\infty$ for all $E \in \mathcal{A}$.

⑦ If u is LEBESGUE INTEGRABLE then

$$\mu(\{x \in X : |u(x)| = +\infty\}) = 0.$$

(8) If u, v are integrable and $\alpha, \beta \in \mathbb{R}$ then $\alpha u + \beta v$ is integrable, and

$$\int_X (\alpha u + \beta v)\, d\mu = \alpha \int_X u \, d\mu + \beta \int_X v \, d\mu .$$

(9) If u, v are integrable and $u = v$ μ-a.e. in X, then

$$\int_X u \, d\mu = \int_X v \, d\mu$$

(10) If u is integrable then

$$\left| \int_X u \, d\mu \right| \leq \int_X |u| \, d\mu$$

UNFORGETTABLE THEOREMS

We recall a few theorems concerning the Lebesgue integral:

THEOREM 6.5 (MONOTONE CONVERGENCE)

Let $(X, \mathcal{A}, \mu)$ be a measurable space, and $u_n : X \to [0, +\infty]$ s.t.

1. u_n is measurable $\forall n \in \mathbb{N}$
2. $0 \leq u_1(x) \leq u_2(x) \leq \dots$ for all $x \in X$
3. $u_n(x) \to u(x)$ as $n \to +\infty$ for all $x \in X$

Then

$$\lim_{n \to +\infty} \int_X u_n \, d\mu = \int_X u \, d\mu$$

THEOREM 6.6 (FATOU'S LEMMA)

Let $(X, \mathcal{A}, \mu)$ be a measurable space. If $u_n : X \to [0, +\infty]$ is a sequence of measurable functions, then

$$\int_X \liminf_{n \to +\infty} u_n(x) \, d\mu(x) \leq \liminf_{n \to +\infty} \int_X u_n(x) \, d\mu(x)$$

THEOREM 6.7 (DOMINATED CONVERGENCE)

Let $(X, \mathcal{A}, \mu)$ be a measurable space and $u_n : X \to [-\infty, +\infty]$ a sequence of measurable functions. Suppose that:

1. $u_n(x) \to u(x)$ as $n \to +\infty$, for μ-a.e. $x \in X$
2. $\exists\, v$ Lebesgue integrable such that

$$|u_n(x)| \leq v(x), \quad \forall n \in \mathbb{N} \text{ and } \mu\text{-a.e. } x \in X.$$

Then u is Lebesgue integrable and

$$\lim_{n \to +\infty} \int_X |u_n - u| \, d\mu = 0$$

THEOREM 6.8 (JENSEN'S INEQUALITY)

Let $(X, \mathcal{A}, \mu)$ be measurable space, with $\mu(X) = 1$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex. For all $u : X \to \mathbb{R}$ integrable we have

$$\varphi\left(\int_X u \, d\mu \right) \leq \int_X \varphi \circ u \, d\mu .$$

Finally we recall FUBINI'S and TONELLI'S THEOREMS. We first need:

PRODUCT MEASURE Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be measurable spaces. On the cartesian product $X_1 \times X_2$ define the PRODUCT σ-ALGEBRA

$$\mathcal{A}_1 \otimes \mathcal{A}_2 := \bigcap \left\{ \mathcal{A} \subseteq \mathcal{P}(X_1 \times X_2) \mid \mathcal{A} \text{ is a } \sigma\text{-algebra}, (E_1 \times E_2) \in \mathcal{A}, \ \forall E_i \in \mathcal{A}_i \right\}$$

Thus $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the smallest σ-algebra on $X_1 \times X_2$ containing all the sets of the form $E_1 \times E_2$ with $E_i \in \mathcal{A}_i$. Whenever μ_1, μ_2 are σ-FINITE, there exists a unique measure $\mu_1 \otimes \mu_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to [0, +\infty]$ such that

$$(\mu_1 \otimes \mu_2)(E_1 \times E_2) = \mu_1(E_1)\, \mu_2(E_2), \qquad \forall E_i \in \mathcal{A}_i$$

(it can be constructed via PROP 6.1 and THM 6.2). The measure $\mu_1 \otimes \mu_2$ is called PRODUCT MEASURE between μ_1 and μ_2.

NOTE For the Lebesgue measure it holds that $\mathcal{L}^{d_1} \otimes \mathcal{L}^{d_2} = \mathcal{L}^{d_1+d_2}$.

THEOREM 6.9 (TONELLI)

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be measurable spaces, with μ_1, μ_2 σ-finite. Let $u : X_1 \times X_2 \to \mathbb{R}$ be measurable WRT $\mathcal{A}_1 \otimes \mathcal{A}_2$, and s.t.

(a) For μ_1-a.e. $x \in X_1$ the map $y \in X_2 \mapsto u(x,y)$ is measurable and it holds

$$\int_{X_2} |u(x,y)| \, d\mu_2(y) < +\infty \quad \text{for } \mu_1\text{-a.e. } x \in X_1$$

(b) $\displaystyle\int_{X_1} \left(\int_{X_2} |u(x,y)| \, d\mu_2(y) \right) d\mu_1(x) < +\infty$

Then u is INTEGRABLE WRT the product measure $\mu_1 \otimes \mu_2$.

THEOREM 6.10 (FUBINI)

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be measurable spaces, with μ_1, μ_2 σ-finite. Let $u: X_1 \times X_2 \to [-\infty, +\infty]$ be measurable WRT $\mathcal{A}_1 \otimes \mathcal{A}_2$ and integrable WRT $\mu_1 \otimes \mu_2$. Then

1. For μ_1-a.e. $x \in X_1$ the map $y \in X_2 \mapsto u(x,y)$ is measurable and

$$\int_{X_2} |u(x,y)| \, d\mu_2(y) < +\infty, \qquad \int_{X_1} \left\{ \int_{X_2} |u(x,y)| \, d\mu_2(y) \right\} d\mu_1(x) < +\infty$$

2. For μ_2-a.e. $y \in X_2$ the map $x \in X_1 \mapsto u(x,y)$ is measurable and

$$\int_{X_1} |u(x,y)| \, d\mu_1(x) < +\infty, \qquad \int_{X_2} \left\{ \int_{X_1} |u(x,y)| \, d\mu_1(x) \right\} d\mu_2(y) < +\infty$$

3. The so-called FUBINI'S FORMULA holds:

$$\int_{X_1 \times X_2} |u(x,y)| \, d(\mu_1 \otimes \mu_2)(x,y) = \int_{X_1} \left\{ \int_{X_2} |u(x,y)| \, d\mu_2(y) \right\} d\mu_1(x)$$

$$= \int_{X_2} \left\{ \int_{X_1} |u(x,y)| \, d\mu_1(x) \right\} d\mu_2(y)$$

$\boxed{L^p \text{ SPACES}}$ Let $(X, \mathcal{A}, \mu)$ be a measurable space. For $p \geq 1$ we set

$$L^p(X, \mu) := \left\{ u : X \to \mathbb{R} \mid u \text{ measurable}, \int_X |u|^p \, d\mu < +\infty \right\}$$

In other words, $u \in L^p(X, \mu)$ iff u is μ-INTEGRABLE.

For the case $p = +\infty$ we have an ad-hoc definition

$$L^\infty(X,\mu) := \{ u : X \to \mathbb{R} \mid u \text{ measurable}, \ \exists\, C>0 \text{ s.t. } |u(x)| \le C \ \ \mu\text{-a.e. in } X \}$$

The condition $|u(x)| \le C$ for μ-a.e. $x \in X$ is called ESSENTIAL BOUNDEDNESS.

WARNING The elements of $L^p(X,\mu)$ and $L^\infty(X,\mu)$ are NOT functions, but classes of equivalence of functions, where the equivalence is

$$u \sim v \iff u(x) = v(x) \ \text{ for } \mu\text{-a.e. } x \in X$$

This is not an issue, since in this case $\int_X u \, d\mu = \int_X v \, d\mu$. Therefore $L^p(X,\mu)$ and $L^\infty(X,\mu)$ have to be understood as QUOTIENT SPACES WRT $\sim$

THEOREM 6.11 Let $1 \le p \le +\infty$ and define the CONJUGATE EXPONENT $p' := \frac{p}{p-1}$. If $u \in L^p(X,\mu)$, $v \in L^{p'}(X,\mu)$ then

$$uv \in L^1(X,\mu) \quad \text{and} \quad \underbrace{\|uv\|_1 \le \|u\|_p \, \|v\|_{p'}}_{\text{HÖLDER'S INEQUALITY}}$$

THEOREM 6.12 $L^p(X,\mu)$, $L^\infty(X,\mu)$ are Banach spaces with the norms

$$\|u\|_p := \left(\int_X |u|^p \, d\mu \right)^{1/p}, \quad u \in L^p(X,\mu)$$

$$\|u\|_\infty := \inf\{ C : |u(x)| \le C \ \ \mu\text{-a.e. in } X \}, \quad u \in L^\infty(X,\mu)$$

Moreover $L^2(X,\mu)$ is a Hilbert space with inner product

$$\langle u, v \rangle := \int_X uv \, d\mu, \quad u, v \in L^2(X,\mu)$$

A standard corollary of the proof of THEOREM 6.12 is the following.

PROPOSITION 6.13 Let $\{u_n\} \subseteq L^p(X,\mu)$ and suppose $u_n \to u$ strongly. Then there exist a subsequence u_{n_k} and $h \in L^p(X,\mu)$ s.t.

(a) $u_{n_k}(x) \to u(x)$ as $k \to +\infty$ for μ-a.e. $x \in X$

(b) $\sup_k |u_{n_k}(x)| \leq h(x)$ for μ-a.e. $x \in X$

THEOREM 6.14 (DUALITY)

Let $1 < p < +\infty$. Then $L^p(X,\mu)^* \cong L^{p'}(X,\mu)$, with isometry

$$
\begin{aligned}
L^{p'}(X,\mu) &\longrightarrow L^p(X,\mu)^* \\
u &\longmapsto \left(v \mapsto \int_X u v \, d\mu \right)
\end{aligned}
$$

In particular, as $(p')' = p$, we have that $L^p(X,\mu)$ is REFLEXIVE.
Also $L^1(X,\mu)^* \cong L^\infty(X,\mu)$.

WARNING It is NOT TRUE that $L^\infty(X,\mu)^* \cong L^1(X,\mu)$.

We now recall a result about SEPARABILITY of L^p spaces. We need first the following definition

SEPARABLE MEASURE SPACE

Let $(X,\mathcal{A})$ be a SEPARABLE measure space, i.e., $\exists \{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ s.t. $\sigma(\{E_n\}) = \mathcal{A}$, where

$$\sigma(\{E_n\}) := \bigcap \{ \mathcal{M} \mid \mathcal{M} \text{ is } \sigma\text{-algebra on } X, \ \{E_n\} \subseteq \mathcal{M} \},$$

i.e., $\sigma(\{E_n\})$ is the smallest σ-algebra on X which contains $\{E_n\}$.

EXAMPLE

- $\mathbb{R}^d$ is separable with the Borel σ-algebra.
- $(\mathbb{R}^d, \mathcal{L}^*)$ is separable, where $\mathcal{L}^*$ is the σ-algebra of Lebesgue measurable sets
- (X, d) separable metric space, τ_d topology induced by d. Then $(X, \sigma(\tau_d))$ is a separable measure space.

THEOREM 6.15 (SEPARABILITY) Let $(X, \mathcal{A}, \mu)$ be a SEPARABLE measure space. Then $L^p(X, \mu)$ equipped with the standard norm is SEPARABLE, for all $1 \leq p < +\infty$.
The space $L^\infty(X, \mu)$ is in general NOT separable.

We summarize the above results in a table

	REFLEXIVE	SEPARABLE	DUAL SPACE
L^p with $1<p<+\infty$	YES	YES	$L^{p'}$
L^1	NO	YES	L^∞
L^∞	NO	NO	Strictly bigger than L^1

Finally we conclude with a useful density result:

THEOREM 6.16 Consider $(\mathbb{R}^d, \mathcal{L}^*, \mathcal{L}^d)$, where $\mathcal{L}^*$ is the LEBESGUE σ-algebra and $\mathcal{L}^d$ is the d-dimensional LEBESGUE MEASURE. Let $1 \leq p < +\infty$. Then $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, i.e.,

$$\forall\, u \in L^p(\mathbb{R}^d),\ \forall\, \varepsilon > 0,\ \exists\, v \in C_c(\mathbb{R}^d) \text{ s.t. } \|u - v\|_p \leq \varepsilon.$$

STRONG COMPACTNESS IN L^p

We conclude with a STRONG COMPACTNESS criterion for L^p spaces. To this end, given $f: \mathbb{R}^d \to \mathbb{R}$, $h \in \mathbb{R}^d$, we define the SHIFT of f by h as the function $\tau_h f: \mathbb{R}^d \to \mathbb{R}$

$$(\tau_h f)(x) := f(x+\tau), \qquad \forall x \in \mathbb{R}^d.$$

THEOREM 6.17 (FRÉCHET - KOLMOGOROV)

Let $1 \leq p < +\infty$ and $A \subseteq L^p(\mathbb{R}^d)$. For a measurable set $\Omega \subseteq \mathbb{R}^d$ with finite measure, we denote by $A_{|\Omega}$ the restrictions to Ω of the functions in A, i.e.,

$$A_{|\Omega} = \{ v: \Omega \to \mathbb{R} \mid \exists u \in A \text{ s.t. } v = u_{|\Omega} \}.$$

Assume that

1. A is BOUNDED: i.e., $\exists M > 0$ s.t. $\|u\|_{L^p(\mathbb{R}^d)} \leq M$, $\forall u \in A$

2. A is EQUI-INTEGRABLE: i.e.,

$$\lim_{|h| \to 0} \left\{ \sup_{u \in A} \| \tau_h u - u \|_{L^p(\mathbb{R}^d)} \right\} = 0.$$

Then the closure of $A_{|\Omega}$ in $L^p(\Omega)$ is COMPACT, i.e., if $\{u_n\} \subseteq \overline{A_{|\Omega}}$, $\exists \bar{u} \in \overline{A_{|\Omega}}$ and a subsequence n_k such that

$$u_{n_k} \to \bar{u} \quad \text{as } k \to +\infty \quad \text{strongly in } L^p(\Omega)$$

OTHER MEASURE THEORETIC RESULTS

THEOREM 6.18 (ABSOLUTE CONTINUITY OF LEBESGUE INTEGRAL)

Let $(X, \mathcal{A}, \mu)$ be a measure space and $u \in L^1(X;\mu)$. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\mu(E) < \delta \quad \Rightarrow \quad \left| \int_E u \, d\mu \right| < \varepsilon .$$

THEOREM 6.19 (EGOROFF)

Let $(X, \mathcal{A}, \mu)$ be a measure space, with $\mu(X) < +\infty$.
Suppose $f_n : X \to \mathbb{R}$ is a sequence s.t.

$$f_n \to f \quad \text{a.e. in } X$$

Then $\forall \varepsilon > 0$, $\exists$ $E_\varepsilon \in \mathcal{A}$ s.t. $\mu(E_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on $X \setminus E_\varepsilon$, i.e.

$$\lim_{n \to +\infty} \sup_{x \in X \setminus E_\varepsilon} |f_n(x) - f(x)| = 0 .$$

THEOREM 6.20 (LUSIN)

Let $\Omega \subseteq \mathbb{R}^d$ with $|\Omega| < +\infty$. Let $u : \Omega \to \mathbb{R}$ be Lebesgue measurable.
Then $\forall \varepsilon > 0$, $\exists K \subseteq \Omega$ compact such that

$$|\Omega \setminus K| < \varepsilon \quad \text{and} \quad u_{|K} \text{ is CONTINUOUS.}$$