LESSON 5 - 14 APRIL 2021

4. THE EULER - LAGRANGE EQUATION

After the many examples seen so far, we look at the general theory for the minimization of integral functionals

$$F(u) := \int_a^b L(x, u(x), \dot{u}(x))\, dx$$

where $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x, s, p)$, is the LAGRANGIAN and $u : [a,b] \to \mathbb{R}$. We want to make sufficient assumptions on L so that F admits the first variation δF in some appropriate domain of definition. Specifically, we have:

THEOREM 4.1 Suppose that $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and continuously partially differentiable WRT to the variables s, p.
Let $X \subseteq C^1([a,b])$ be an affine space, with reference vector space $V \subseteq C^1[a,b]$.
Define $F : X \to \mathbb{R}$ by setting

$$F(u) := \int_a^b L(x, u(x), \dot{u}(x))\, dx$$

Then F is Gâteaux differentiable at all points $u \in X$ and all directions $v \in V$, with

$$F_g'(u)(v) = \int_a^b L_s(x, u, \dot{u})\, v + L_p(x, u, \dot{u})\, \dot{v}\, dx$$

with $L_s := \partial_s L$, $L_p := \partial_p L$. In particular $\delta F(u, v)$ exists, with

$$\boxed{\delta F(u, v) = \int_a^b L_s(x, u, \dot{u})\, v + L_p(x, u, \dot{u})\, \dot{v}\, dx}$$

NOTE: Here $C^1[a,b]$ is equipped with the norm $\|u\| := \|u\|_\infty + \|u'\|_\infty$.

Proof Let $u \in X$, $v \in V$. As X is affine space over V, then $u + tv \in X$, $\forall t \in \mathbb{R}$. Then

$$(*) \quad \frac{F(u+tv) - F(u)}{t} = \int_a^b \underbrace{\frac{L(x, u+tv, \dot{u}+t\dot{v}) - L(x,u,\dot{u})}{t}}_{=: \Lambda(t,x)} dx$$

Now suppose $|t| \le \varepsilon$. Then

$$\Lambda(t,x) = \frac{1}{t}\int_0^t \left\{ \frac{d}{d\tau} L(x, u(x) + \tau v(x), \dot{u}(x) + \tau \dot{v}(x)) \right\} d\tau$$

As L diff. in s, p $\rightarrow$
$$= \frac{1}{t}\int_0^t \{ L_s(x, u+\tau v + \dot{u} + \tau\dot{v}) v + L_p(x, u+\tau v, \dot{u}+\tau\dot{v}) \dot{v} \} d\tau$$

Adding and subtracting $\rightarrow$
$$= L_s(x, u, \dot{u}) v + L_p(x,u,\dot{u})\dot{v}$$
$$+ \frac{1}{t}\int_0^t \{ L_s(x, u+\tau v, \dot{u}+\tau\dot{v}) v - L_s(x,u,\dot{u}) v \} d\tau \quad (=: R_1(t,x))$$
$$+ \frac{1}{t}\int_0^t \{ L_p(x, u+\tau v, \dot{u}+\tau\dot{v}) \dot{v} - L_p(x,u,\dot{u})\dot{v} \} d\tau \quad (=: R_2(t,x))$$

Thus, by (*),

$$\frac{F(u+tv) - F(u)}{t} = \int_a^b \Lambda(t,x)\, dx$$
$$= \int_a^b L_s(x,u,\dot{u}) v + L_p(x,u,\dot{u})\dot{v}\, dx + \int_a^b R_1(t,x)\, dx + \int_a^b R_2(t,x) dx$$

To see that F is Gâteaux diff it is sufficient to show that

$$\lim_{t\to 0} \int_a^b R_j(t,x) dx = 0, \quad \text{for } j = 1,2.$$

To this end, notice that, as $u, v \in C^1[a,b]$, then

$$K := \{ (x, u(x) + \tau v(x), \dot{u}(x) + \tau \dot{v}(x)) \mid x \in [a,b], |\tau| \leq \varepsilon/2 \}$$

is compact in $[a,b] \times \mathbb{R} \times \mathbb{R}$. As L_s is continuous on $[a,b] \times \mathbb{R} \times \mathbb{R}$, then in particular it is uniformly continuous on K (continuous on compact $\Rightarrow$ U.C.)
Then $\forall \tilde{\varepsilon} > 0$, $\exists \delta > 0$ s.t.

$$\left| L_s(x, u(x) + \tau v(x), \dot{u}(x) + \tau \dot{v}(x)) - L_s(x, u(x), \dot{u}(x)) \right| < \tilde{\varepsilon} \quad (*)$$

for all $x \in [a,b]$ and $|\tau| \leq \frac{\varepsilon}{2}$, such that $|\tau| (|v(x)| + |\dot{v}(x)|) < \delta$.
The last condition is fullfilled for τ s.t.

$$|\tau| < \min \left\{ \frac{\varepsilon}{2}, \frac{\delta}{\|v\|} \right\} \quad (B)$$

Therefore let $\hat{\varepsilon} > 0$ be arbitrary and fix $\tilde{\varepsilon} > 0$ s.t.

$$\tilde{\varepsilon} < \frac{\hat{\varepsilon}}{\|v\|} \quad (**)$$

Let also $\tilde{\delta} := \min \{ \varepsilon/2, \frac{\delta}{\|v\|} \}$. Then for $|t| < \delta$ we have

$$|R_1(t,x)| \leq \frac{1}{|t|} \int_0^t | L_s(x, u + \tau v, \dot{u} + \tau \dot{v}) - L_s(x, u, \dot{u}) | \, d\tau \, |v(x)|$$

$$\left(\begin{array}{c} \text{by } (*) \text{, as} \\ |\tau| \leq |t| < \delta \text{, so} \\ \text{that (B) holds} \end{array} \right) \leq \frac{\|v\|}{|t|} \int_0^t \tilde{\varepsilon} \, d\tau = \|v\| \, \tilde{\varepsilon} < \hat{\varepsilon} \quad (\text{by } (**))$$

As $\hat{\varepsilon}$ is arbitrary, and $\tilde{\delta}$ does not depend on x, we conclude

$$\lim_{t \to 0} \sup_{x \in [a,b]} |R_1(t,x)| = 0.$$

By similar arguments also $\lim_{t\to 0} \sup_{x\in[a,b]} |R_2(t,x)| = 0$.

Then $\int_a^b R_i(t,x)\,dx \to 0$ as $t \to 0$. Taking the limit in

$$\frac{F(u+tv)-F(u)}{t} = \int_a^b L_s(x,u,\dot{u})v + L_p(x,u,\dot{u})\dot{v}\,dx + \int_a^b R_1(t,x)\,dx + \int_a^b R_2(t,x)\,dx$$

yields that

$$F_g'(u)(v) = \int_a^b L_s(x,u,\dot{u})v + L_p(x,u,\dot{u})\dot{v}\,dx,$$

as claimed. Now just recall that for affine spaces which are also normed,

$$\delta F(u,v) := \lim_{t\to 0} \frac{F(u+tv)-F(u)}{t}$$

(see REMARK 2.5). This concludes. □

DEFINITION 4.2 In the setting of THEOREM 4.1, we call

$$(*) \qquad \boxed{\delta F(u,v) = \int_a^b L_s(x,u,\dot{u})v + L_p(x,u,\dot{u})\dot{v}\,dx}$$

the FIRST INTEGRAL FORM of the FIRST VARIATION.

CASE OF DIRICHLET BOUNDARY CONDITIONS

Assume $L : [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ continuous, and continuously diff. in s,p. Let

$$X = \{ u \in C^1[a,b] \mid u(a) = \alpha,\ u(b) = \beta \}$$

which is an affine space over

$$V = \{ v \in C^1[a,b] \mid v(a) = v(b) = 0 \}.$$

Then we can apply THEOREM 4.1, and $\delta F(u,v)$ is given by $(*)$.

Assume also that

$$L \in C^2([a,b]\times\mathbb{R}\times\mathbb{R}), \qquad u \in C^2[a,b] \cap X$$

Then the second term in (*) can be integrated by parts:

$$\int_a^b L_p(x,u,\dot u)\,\dot v\,dx = L_p(x,u,\dot u)\,v\Big|_a^b - \int_a^b (L_p(x,u,\dot u))'\,v(x)\,dx$$

$$= -\int_a^b (L_p(x,u,\dot u))'\,v(x)\,dx \qquad (\text{as } v(a)=v(b)=0)$$

Therefore (*) reads

(**) $$\boxed{\delta F(u,v) = \int_a^b \left\{ L_s(x,\dot u,\dot u) - (L_p(x,u,\dot u))' \right\} v(x)\,dx}$$

Note that, in the above assumptions, we can explicitly compute

$$(L_p(x,u,\dot u))' = L_{px}(x,u,\dot u) + L_{ps}(x,u,\dot u)\,\dot u + L_{pp}(x,u,\dot u)\,\ddot u$$

DEFINITION 4.3 (**) is called the SECOND INTEGRAL FORM of the FIRST VARIATION

Assume in addition that u minimizes F over X.

Then by REMARK 3.7 we know that $\delta F(u,v)=0$, $\forall v \in V$. Note that $C_c^\infty(a,b) \subseteq V$. Hence we can apply the FLCV (LEMMA 3.4) to (**) (equated to zero), and obtain

(***) $$\boxed{[L_p(x,u,\dot u)]' = L_s(x,u,\dot u)}$$

Note that, in addition to (***), u satisfies also the DIRICHLET BC imposed in X, that is,

$$u(a)=\alpha, \quad u(b)=\beta$$

DEFINITION 4.4 (***) is called **EULER-LAGRANGE EQUATION in DIFFERENTIAL FORM.**

We therefore have proven the following theorem.

THEOREM 4.5 Let $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and continuously partially differentiable in s, p.
Define

$$X := \{ u \in C^1[a,b] \mid u(a) = \alpha, u(b) = \beta \}$$

$$V := \{ v \in C^1[a,b] \mid v(a) = v(b) = 0 \}$$

Define the functional $F : X \to \mathbb{R}$ s.t.

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx$$

(1) If $u \in X$ minimizes F over X, then u solves the **ELE in INTEGRAL FORM**:

$$\int_a^b L_s(x, u, \dot{u})\, v + L_p(x, u, \dot{u})\, \dot{v} \; dx = 0$$

for all $v \in V$.

(2) Assume in addition $L \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$.
If $u \in X \cap C^2[a,b]$ minimizes F over X, then u solves the **ELE in DIFFERENTIAL FORM**:

$$\begin{cases} \dfrac{d}{dx}\Big[L_p(x, u(x), \dot{u}(x)) \Big] = L_s(x, u(x), \dot{u}(x)), & \forall x \in (a,b) \\ u(a) = \alpha, \quad u(b) = \beta \end{cases}$$

THE CASE OF NEUMANN BOUNDARY CONDITIONS

Again, suppose $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and continuously diff. in s, p. Define

$$X = \{ u \in C^1[a,b] \mid u(a) = \alpha \}$$

which is affine over

$$V = \{ v \in C^1[a,b] \mid v(a) = 0 \}.$$

We can then apply THEOREM 4.1 to obtain the FIRST INTEGRAL FORM of the FIRST VARIATION:

$$(*) \qquad \boxed{\delta F(u,v) = \int_a^b L_s(x,u,\dot{u})\, v + L_p(x,u,\dot{u})\, \dot{v} \; dx}$$

Assume in addition that

$$L \in C^2([a,b] \times \mathbb{R} \times \mathbb{R}), \qquad u \in C^2[a,b] \cap X.$$

Then the second term in $(*)$ can be integrated by parts

$$\int_a^b L_p(x,u,\dot{u})\, \dot{v}\, dx = L_p(x,u,\dot{u})\, v \Big|_a^b - \int_a^b (L_p(x,u,\dot{u}))'\, v(x)\, dx$$

$$= L_p(b, u(b), \dot{u}(b))\, v(b) - \int_a^b \left[L_p(x,u,\dot{u}) \right]' v \, dx$$

obtaining the SECOND INTEGRAL FORM of the FIRST VARIATION:

$$(**) \qquad \boxed{\delta F(u,v) = \int_a^b \left\{ L_s(x,u,\dot{u}) - \left[L_p(x,u,\dot{u}) \right]' \right\} v \, dx + L_p(b, u(b), \dot{u}(b))\, v(b)}$$

Assume now that u is also a minimizer. Then by REMARK 3.7 we have $\delta F(u, v) = 0$, $\forall v \in V$. In particular we can test for

$$v \in V \quad \text{s.t} \quad v(b) = 0$$

to obtain

$$\int_a^b \{ L_s(x, u, \dot{u}) - [L_p(x, u, \dot{u})]' \} v \, dx = 0, \quad \forall v \in C^1[a,b] \text{ s.t. } v(a) = v(b) = 0.$$

Then by FLCV we obtain the EULER-LAGRANGE EQUATION in DIFF. FORM:

$$(***) \qquad \boxed{[L_p(x, u, \dot{u})]' = L_s(x, u, \dot{u})}$$

Now, the first boundary condition to pair to $(***)$ is already given in X:

$$u(a) = \alpha$$

For the second BC, just test $(**)$ against $v \in V$ s.t. $v(b) \neq 0$, and recall $(***)$, to get

$$L_p(b, u(b), \dot{u}(b)) v(b) = 0, \quad \forall v \in V \quad \xrightarrow{\substack{\text{Taking } v \in V \\ \text{s.t. } v(b) \neq 0}} \quad L_p(b, u(b), \dot{u}(b)) = 0$$

which is a NEUMANN BOUNDARY CONDITION.

NOTE If we took $X = V = C^1[a,b]$ in the above example, we would have obtained that minimizers $u \in C^2[a,b] \cap X$ satisfy $(***)$ with two NEUMANN BC

$$L_p(a, u(a), \dot{u}(a)) = L_p(b, u(b), \dot{u}(b)) = 0$$

To summarize, we have proven the following Theorem:

THEOREM 4.6 Let $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and continuously partially differentiable WRT s, p.
Define sets

$$X := \{ u \in C^1[a,b] \mid u(a) = \alpha \}$$

$$V := \{ v \in C^1[a,b] \mid v(a) = 0 \}$$

Define $F : X \longrightarrow \mathbb{R}$ by

$$F(u) := \int_a^b L(x, u, \dot{u})\, dx$$

(1) Suppose u minimizes F over X. Then u solves the ELE in INTEGRAL FORM:

$$\int_a^b L_s(x, u, \dot{u})\, v + L_p(x, u, \dot{u})\, \dot{v} \, dx = 0 , \quad \forall v \in V$$

(2) Suppose in addition $L \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$, and that $u \in X \cap C^2[a,b]$ minimizes F over X. Then u solves ELE in DIFFERENTIAL FORM:

$$\begin{cases} \dfrac{d}{dx}\Big[L_p(x, u(x), \dot{u}(x)) \Big] = L_s(x, u(x), \dot{u}(x)) , & \forall x \in (a,b) \\ u(a) = \alpha , \quad L_p(b, u(b), \dot{u}(b)) = 0 \end{cases}$$

ELE IN ERDMANN FORM

Consider the special case of Lagrangians not depending on x, i.e.,

$$F(u) = \int_a^b L(u, \dot{u})\, dx \quad , \quad L : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$$

with $F : X \to \mathbb{R}$, $X \subseteq C^1[a,b]$ affine space over $V \subseteq C^1[a,b]$.

As done previously, if $L \in C^2(\mathbb{R}\times\mathbb{R})$ and $u \in C^2[a,b] \cap X$ minimizes F over X, the ELE reads

$$(*) \quad \left[L_p(u, \dot{u}) \right]' = L_s(u, \dot{u}) \quad , \quad \forall x \in (a,b)$$

Multiplying by $\dot{u}$ yields

$$(**) \quad \left[L_p(u, \dot{u}) \right]' \dot{u} = L_s(u, \dot{u})\, \dot{u}$$

Now the LHS is

$$\left[L_p(u, \dot{u}) \right]' \dot{u} = \left[L_p(u, \dot{u})\, \dot{u} \right]' - L_p(u, \dot{u})\, \ddot{u}$$

so that, from $(**)$

$$\left[L_p(u, \dot{u})\, \dot{u} \right]' \overset{\text{by } (**)}{=} L_s(u, \dot{u})\, \dot{u} + L_p(u, \dot{u})\, \ddot{u} \overset{\text{by direct calculation}}{=} \left[L(u, \dot{u}) \right]'$$

Therefore

$$\boxed{L_p(u, \dot{u})\, \dot{u} = L(u, \dot{u}) + \text{constant}}$$

ELE in ERDMANN FORM

REMARK 4.5 ELE and ELE-ERDMANN are <u>not</u> equivalent. It holds:

(1) If u satisfies ELE $\Rightarrow$ u satisfies ELE-ERDMANN

(we just proved this)

(2) If u satisfies ELE-ERDMANN $\Rightarrow$ u satisfies ELE in the points $x \in [a,b]$ s.t. $\dot{u}(x) \neq 0$

(To show this, just go backwards in the above calculation)

ELE FOR GENERAL LAGRANGIANS

HIGHER ORDER

$X \subseteq C^k[a,b]$ affine space over $V \subseteq C^k[a,b]$, $L: [a,b] \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$

$L = L(x, s, p_1, \dots, p_k)$, $F: X \to \mathbb{R}$ defined by:

$$F(u) := \int_a^b L(x, u, \dot{u}, \ddot{u}, \dots, u^{(k)})\, dx$$

Assume L is continuous and continuously differentiable WRT $s, p_1, \dots, p_k$.

Analogously to THEOREM 4.1, one can compute the Gâteaux derivative of F and obtain the FIRST INTEGRAL FORM of the FIRST VARIATION

$$(*) \quad \boxed{\delta F(u,v) = \int_a^b L_s(x, u, \dots, u^{(k)})\, v + \sum_{i=1}^{k} L_{p_i}(x, u, \dots, u^{(k)})\, v^{(i)}\, dx}$$

Assume now that $L \in C^2([a,b] \times \mathbb{R} \times \mathbb{R}^k)$, $u \in C^{k+1}[a,b]$, and $v \in V$ is s.t. $v^{(i)}(a) = v^{(i)}(b) = 0$ for all $i = 0, .., k-1$. Integrating (*) by parts we get the SECOND INTEGRAL FORM of the FIRST VARIATION

$$\boxed{\delta F(u,v) = \int_a^b \left\{ L_s(x,u,..,u^{(k)}) + \sum_{i=1}^{k} (-1)^i \frac{d^i}{dx^i} L_{p_i}(x,u,..,u^{(k)}) \right\} v \, dx}$$

Finally, if in addition u is a minimizer, then $\delta F(u,v) = 0$, and by the FLCV we get the ELE in DIFFERENTIAL FORM

$$\boxed{\sum_{i=1}^{k} (-1)^{i+1} \frac{d^i}{dx^i} L_{p_i}(x,u,..,u^{(k)}) = L_s(x,u,..,u^{(k)})}, \quad \forall x \in (a,b)$$

MORE UNKNOWNS

$X \subseteq C^1[a,b]$ affine space over $V \subseteq C^1[a,b]$, $L : [a,b] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$,

$L = L(x, s_1, .., s_k, p_1, .., p_k)$, $F : \underbrace{X \times \cdots \times X}_{k \text{ times}} \to \mathbb{R}$ defined by

$$F(u_1, .., u_k) := \int_a^b L(x, u_1, .., u_k, \dot{u}_1, .., \dot{u}_k) \, dx$$

Assume L is continuous and continuously differentiable in $s_1, .., s_k, p_1, .., p_k$.

Analogously to THEOREM 4.1, one can compute the Gâteaux derivative of F and obtain the FIRST INTEGRAL FORM of the FIRST VARIATION

$$\boxed{\delta F(u,v) = \int_a^b \sum_{i=1}^k \left[L_{s_i}(x,u,\dot{u})\, v_i + L_{p_i}(x,u,\dot{u})\, \dot{v}_i \right] dx} \quad (*)$$

where $u = (u_1, \dots, u_k) \in X^k$, $v = (v_1, \dots, v_k) \in X^k$.

Suppose in addition that $L \in C^2([a,b] \times \mathbb{R}^k \times \mathbb{R}^k)$, $u_i \in C^2[a,b] \cap X$ and that $v_i \in V$ are s.t. $v_i(a) = v_i(b) = 0$, for all $i = 1, \dots, k$. Then we can integrate $(*)$ by parts to get the SECOND INTEGRAL FORM of the FIRST VARIATION

$$\boxed{\delta F(u,v) = \int_a^b \sum_{i=1}^k \left[L_{s_i}(x,u,\dot{u}) - L_{p_i}(x,u,\dot{u})' \right] v_i \, dx}$$

Finally, taking $u \in X^k$ minimum of F and $v_1 \in C_c^\infty(a,b)$, $v_2 = v_3 = \dots = v_k = 0$ and applying FLCV, we get

$$L_{p_1}(x,u,\dot{u})' = L_{s_1}(x,u,\dot{u})$$

Similarly, by taking the other components of v to be zero, except for one, we obtain the ELE in DIFFERENTIAL FORM

$$\boxed{L_{p_i}(x,u,\dot{u})' = L_{s_i}(x,u,\dot{u}), \quad i = 1, \dots, k} \quad , \ \forall x \in (a,b)$$

which in this case is a SYSTEM of k ODEs of ORDER 2.