LESSON 4 - 24 MARCH 2021

3. FUNDAMENTAL LEMMAS

We now prove two fundamental Lemmas which will be ubiquitous throughout the course (we already used one of them after in the example of F, right before PROPOSITION 2.9).

DEFINITION 3.1 Let $u : (U \subseteq \mathbb{R}) \to \mathbb{R}$. The **SUPPORT** of u is the set

$$\operatorname{supp} u := \overline{\{ x \in U \mid u(x) \neq 0 \}}$$

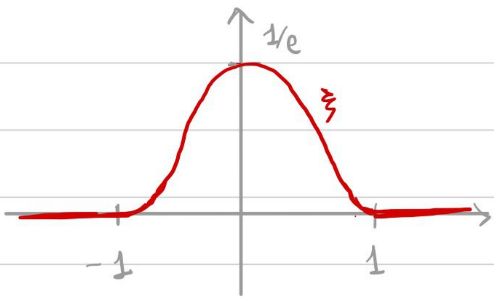
We define the space of SMOOTH COMPACTLY SUPPORTED functions on (a,b) as

$$C_c^\infty(a,b) := \{ u \in C^\infty(a,b) \mid \operatorname{supp} u \text{ is COMPACT} \}$$

In other words, $u \in C_c^\infty(a,b)$ iff $\exists \; [c,d] \subseteq (a,b)$ s.t. $\operatorname{supp} u \subseteq [c,d]$, i.e., $u \equiv 0 \quad (a,b) \smallsetminus [c,d]$.

REMARK 3.2 We can construct $u \in C_c^\infty(a,b)$ having PRESCRIBED support in some interval $[c,d] \subseteq (a,b)$, and having the same sign, i.e., either $u \geq 0$ or $u \leq 0$.

To do that, consider the **BUMP FUNCTION**



$$\xi(x) := \begin{cases} \exp\left(- \dfrac{1}{1-x^2} \right), & \text{if } |x| < 1 \\ 0 & , \text{if } |x| \geq 1 \end{cases}$$

Then $\xi \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} \xi \subseteq [-1,1]$ and $\xi > 0$ in $(-1,1)$.

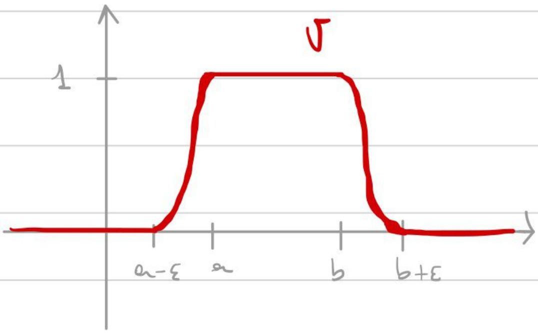
For $x_0 \in \mathbb{R}, r > 0$ fixed define

$$(*) \qquad u(x) := \xi\left(\frac{x - x_0}{r} \right)$$

Then $u \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} u \subseteq [x_0 - r, x_0 + r]$ and $u > 0$ in $(x_0 - r, x_0 + r)$

(To get $u < 0$ just consider $-\xi$ in the definition $(*)$)

REMARK 3.3 Using the function ξ at REMARK 3.2 and CONVOLUTIONS, it is possible to construct $v \in C_c^\infty(\mathbb{R})$ such that $0 \leq v \leq 1$ and



$$v(x) = \begin{cases} 1 & x \in [a,b] \\ 0 \quad \text{if} & x \notin [a-\varepsilon, b+\varepsilon] \end{cases}$$

where a, b and $\varepsilon > 0$ can be chosen arbitrarily. Such v is called CUT-OFF function

(We omit the proof of this fact for the moment. It will be left as an exercise in the EXERCISES COURSE).

LEMMA 3.4 (FUNDAMENTAL LEMMA OF CALCULUS OF VARIATIONS) (FLCV)

Let $f : [a,b] \to \mathbb{R}$ be continuous. Assume that

$$\int_a^b f(x)\, v(x)\, dx = 0, \qquad \forall\, v \in C_c^\infty(a,b)$$

Then $f \equiv 0$.

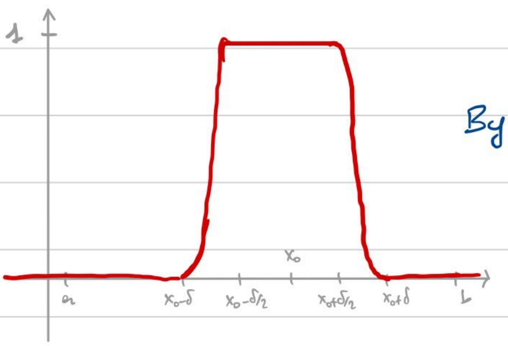
We give 2 proofs of this Lemma, to show different and interesting techniques;

PROOF 1 OF LEMMA 3.4 (By contradiction)

Assume by contradiction that $f \not\equiv 0$. Then WLOG $\exists\, x_0 \in (a,b)$ such that $f(x_0) > 0$. By continuity also $\exists\, \delta > 0$ s.t.

$$f(x) \geq \frac{f(x_0)}{2}, \quad \forall\, x \in (x_0 - \delta, x_0 + \delta) \subseteq [a,b]$$

By REMARK 3.3 $\exists\, v \in C_c^\infty(\mathbb{R})$ s.t. $0 \leq v \leq 1$ and



$$v(x) = \begin{cases} 1 & \text{for } x \in [x_0 - \delta/2, x_0 + \delta/2] \\ 0 & \text{for } x \notin [x_0 - \delta, x_0 + \delta] \end{cases}$$

Thus by assumption we have

$$\int_a^b f(x)\, v(x)\, dx = 0.$$

On the other hand,

$$\int_a^b f(x)\, v(x)\, dx = \int_{x_0-\delta}^{x_0+\delta} f(x)\, v(x)\, dx \geq \int_{x_0-\delta/2}^{x_0+\delta/2} f(x)\, v(x)\, dx \geq \frac{f(x_0)}{2}\,\delta > 0$$

As $v=0$ outside of $[x_0-\delta, x_0+\delta]$

As $v \geq 0$ always, while $f \geq \frac{f(x_0)}{2} > 0$ in $[x_0-\delta, x_0+\delta]$

Since $v \equiv 1$ and $f(x) \geq f(x_0)/2$ here

which is a contradiction. $\square$

Before proceeding with the second proof of LEMMA 3.4, we make the following remark (a proof of which is left for the exercises course)

REMARK 3.5 Let $v : [a,b] \to \mathbb{R}$ continuous. There exists a sequence $\{v_n\} \subseteq C_c^\infty(a,b)$ s.t.

1) $\{v_n\}$ is uniformly bounded, i.e., $\exists\, M > 0$ s.t.

$$\sup_n \| v_n \|_\infty \leq M$$

2) For each $K \subseteq [a,b]$ compact we have that $v_n \to v$ uniformly on K, that is,

$$\lim_{n \to +\infty} \sup_{x \in K} | v_n(x) - v(x) | = 0.$$

PROOF 2 OF LEMMA 3.4 (By Density)

We claim the following

$(*)$ $\displaystyle\int_a^b f(x)\, v(x) = 0, \ \forall\, v \in C_c^\infty(a,b) \implies \int_a^b f(x)\, v(x) = 0, \ \forall\, v \in C(a,b)$

Notice that if (*) holds then the thesis of Lemma 3.4 follows: indeed, as we are assuming that f satisfies

$$\int_a^b f(x)\,v(x)\,dx = 0 \quad , \forall v \in C_c^\infty(a,b) \,,$$

then by (*) we get that

$$\int_a^b f(x)\,v(x)\,dx = 0 \quad , \forall v \in C(a,b) \,.$$

Thus we can choose $v = f$ in the above (as f is continuous by assumption) and obtain

$$\int_a^b |f|^2\,dx = 0 \quad \Rightarrow \quad f \equiv 0 \,,$$

which concludes the proof.

Thus, we are left to show (*). To this end, fix $v \in C(a,b)$. By REMARK 3.5 $\exists \{v_n\} \subseteq C_c^\infty(a,b)$ s.t. $\{v_n\}$ is unif. bounded and $v_n \to v$ uniformly on each $K \subset (a,b)$ compact. As v_n is smooth, by assumption we have

$$(**) \qquad \int_a^b f(x)\,v_n(x)\,dx = 0 \quad , \forall \; n \in \mathbb{N} \,.$$

On the other hand, let $K \subset (a,b)$ be compact. Then

$$\left| \int_a^b f v_n\,dx - \int_a^b f v\,dx \right| \leq \|f\|_\infty \int_a^b |v_n - v|\,dx =$$

$$= \|f\|_\infty \left(\int_K |v_n - v|\,dx + \int_{K^c} |v_n - v|\,dx \right) \qquad \left(K^c := (a,b) \setminus K \right)$$

Now the first integral:

$$\int_K |v_n - v|\,dx \leq |K| \sup_{x \in K} |v_n(x) - v(x)| \to 0 \quad \text{as } n \to +\infty$$

by the properties of v_n. For the second integral we have:

$$\int_{K^c} |v_n - v|\,dx \leq |K^c| \left(\|v_n\|_\infty + \|v\|_\infty \right) \leq |K^c| \left(M + \|v\|_\infty \right)$$

In total,

$$\limsup_{n \to +\infty} \left| \int_a^b f v_n\,dx - \int_a^b f v\,dx \right| \leq \|f\|_\infty |K^c| \left(M + \|v\|_\infty \right).$$

Now, remember that $K \subset (a,b)$ is an arbitrary compact set. Thus $|K^c|$ is as small as we wish, from which we infer

$$\int_a^b f v_n\,dx \to \int_a^b f v\,dx \quad \text{as } n \to +\infty$$

Since (**) holds, we conclude that $\int_a^b f v\,dx = 0$, and the CLAIM is proven. $\square$

The second proof immediately suggests possible generalizations of LEMMA 3.4, which will allow us to test f against a smaller set of functions.

REMARK 3.6 Assume that $f \in C(a,b)$ satisfies

$$\int_a^b f(x) v(x)\,dx = 0, \quad \forall v \in V$$

where $V \subset C(a,b)$ is some set. Then

1) By linearity of the integral we have

$$\int_a^b f(x)\, v(x)\, dx = 0 \,, \quad \forall\, v \in \operatorname{span} V$$

2) By a density argument similar to the one of PROOF 2 of LEMMA 3.4 we have

$$\int_a^b f(x)\, v(x)\, dx = 0 \,, \quad \forall\, v \in \overline{V} \,,$$

where the closure is taken WRT the uniform convergence of bounded sequences on compact sets $K \subset (a,b) \setminus \{x_1, .., x_N\}$ where the collection of points $\{x_1, .., x_N\}$ is FINITE.

As a consequence of REMARK 3.6, and following the arguments of PROOF 2 of LEMMA 3.4 we get:

LEMMA 3.7 (Generalized FLCV)

Let $f \in C(a,b)$, $V \subset C(a,b)$ such that $\overline{\operatorname{span} V} = C(a,b)$, where the closure is as in REMARK 3.6 point (2), i.e.,

$$\overline{\operatorname{span} V} := \Big\{ v \in C(a,b) \;\Big|\; \exists\, \{v_n\} \subseteq \operatorname{span} V, \text{ with } \sup_n \|v_n\|_\infty < +\infty \text{ and } v_n \to v \text{ uniformly on each compact } K \subset (a,b) \setminus I \Big\}$$

with $I := \{x_1, .., x_N\}$ is a fixed finite collection of points. Then

$$\int_a^b f(x)\, v(x) = 0 \,, \quad \forall\, v \in V \;\Rightarrow\; f \equiv 0 \,.$$

We now state and prove a second "fundamental" lemma, which again will be very useful in the rest of the course.

LEMMA 3.8 (DU BOIS REYMOND) (DBR Lemma)

Let $f \in C(a,b)$ and assume that

Zero average function

$$(*) \quad \int_a^b f(x) v(x)\, dx = 0, \quad \forall v \in C_c^\infty(a,b) \text{ s.t. } \int_a^b v(x)\, dx = 0.$$

Then $f \equiv c$ for some $c \in \mathbb{R}$.

Proof The idea is to apply the FLCV (LEMMA 3.4). Thus let $v \in C_c^\infty(a,b)$.
It would be nice if we could use

$$\tilde{v}(x) := v(x) - \frac{1}{b-a} \int_a^b v(y)\, dy$$

as a test function in $(*)$, seeing that $\int_a^b \tilde{v}(x)\, dx = 0$. However $\tilde{v}$ is not compactly supported.

To make this attempt rigorous, take $w \in C_c^\infty(a,b)$ s.t. $\int_a^b w(x)\, dx = 1$, and define

$$\varphi(x) := v(x) - w(x) \int_a^b v(y)\, dy$$

Then $\varphi \in C_c^\infty(a,b)$ and $\int_a^b \varphi(x)\, dx = 0$. By using φ as a test function in $(*)$ we get

$$0 = \int_a^b f(x)\varphi(x)\, dx = \int_a^b f(x) v(x)\, dx - \int_a^b f(x) w(x) \left(\int_a^b v(y)\, dy \right) dx$$

$$= \int_a^b f(x) v(x)\, dx - c \int_a^b v(x)\, dx,$$

where $c := \int_a^b f(x) w(x)\, dx$

Thus

$$0 = \int_a^b f(x)v(x)\,dx - c\int_a^b v(x)\,dx$$

$$= \int_a^b \left[f(x) - c\right] v(x)\,dx$$

Since this is true for all $v \in C_c^\infty(a,b)$, by FLCV LEMMA 3.4 we conclude $f - c \equiv 0 \Rightarrow f \equiv c$. □

A simple (but useful) equivalent formulation of the DBR Lemma is the following one.

LEMMA 3.9 (DBR - Second formulation)

Let $f \in C(a,b)$ and assume that

$$(*) \quad \int_a^b f(x)\dot{v}(x)\,dx = 0, \quad \forall v \in C_c^\infty(a,b)$$

Then $f \equiv c$ for some $c \in \mathbb{R}$.

Proof For $v \in C_c^\infty(a,b)$ we have

$$(**) \quad \int_a^b v(x)\,dx = 0 \iff \exists\, w \in C_c^\infty(a,b) \text{ s.t. } \dot{w} = v$$

Indeed, if $w \in C_c^\infty(a,b)$ is s.t. $\dot{w} = v$, then

$$\int_a^b v(x)\,dx = \int_a^b \dot{w}(x)\,dx = w(b) - w(a) = 0 \quad \left(w \text{ is compactly supported}\right)$$

Conversely, assume $\int_a^b v(x)\,dx = 0$, and let $\varepsilon > 0$ be s.t

$\operatorname{supp} v \subset [a+\varepsilon, b-\varepsilon]$ (since v is compactly supported)

For $x \in [a,b]$ define

$$w(x) := \int_a^x v(y)\,dy$$

Then $\dot{w} = v$, and in particular $w \in C^\infty(a,b)$. Moreover

$$w(x) = \int_a^x v(y)\,dy = 0 \quad \text{if} \quad x \in [a, a+\varepsilon]$$

as $v \equiv 0$ in $[a, a+\varepsilon]$, while

We are assuming this

$$w(x) = \int_a^x v(y)\,dy = \int_a^b v(y)\,dy = 0$$

if $x \in [b-\varepsilon, b]$, as the whole support of v is in $[a, b-\varepsilon]$.

Thus (**) is proven. Now assume that (*) holds. Let $v \in C_c^\infty(a,b)$ be such that $\int_a^b v(x)\,dx = 0$. Then by (**) $\exists\, w \in C_c^\infty(a,b)$ s.t. $\dot{w} = v$. Therefore, by (*), we have $\int_a^b f(x)\,\dot{w}(x) = 0$. Then, as $\dot{w} = v$,

$$\int_a^b f(x)\,v(x) = \int_a^b f(x)\,\dot{w}(x) = 0$$

As v is arbitrary, then $f \equiv c$ by DBR LEMMA 3.8. $\square$

As for the FLCV, also in the DBR lemma we can test f against a smaller set of functions, since the DBR can also be proven with a density argument (very similar to PROOF 2 of LEMMA 3.4). Such argument makes use of the following remark (Again, left for the exercise course)

REMARK 3.10 Let $v \in C(a,b)$ with $\int_a^b v(x)\,dx = 0$. Then $\exists$ $\{v_n\} \subseteq C_c^\infty(a,b)$ such that

1) $\sup_n \|v_n\|_\infty \leq M$, for some $M>0$

2) $v_n \to v$ uniformly on compact sets $K \subset (a,b)$

3) $\int_a^b v_n(x)\,dx = 0$, $\forall n \in \mathbb{N}$.

We have the following alternative proof of the DBR LEMMA 3.8.

ALTERNATIVE PROOF OF LEMMA 3.8 (by density)

By proceeding exactly as in PROOF 2 of LEMMA 3.4 (using REMARK 3.10 in place of REMARK 3.5) we can show that

$$\boxed{\begin{gathered}\int_a^b f(x)\,v(x)\,dx = 0, \quad \forall v \in C_c^\infty(a,b) \text{ with } \int_a^b v(x)\,dx = 0 \\ \Downarrow \\ \int_a^b f(x)\,v(x)\,dx = 0, \quad \forall v \in C(a,b) \text{ with } \int_a^b v(x)\,dx = 0\end{gathered}} \quad (*)$$

Now the thesis of LEMMA 3.8 follows immediately by $(*)$. Indeed, assume that $f \in C(a,b)$ is such that

$$\int_a^b f(x)\,v(x)\,dx = 0, \quad \forall v \in C_c^\infty(a,b) \text{ with } \int_a^b v(x)\,dx = 0.$$

As v has zero average, then also $f+c$ for any $c \in \mathbb{R}$ satisfies the above.

Thus, by $(*)$,

$(**)$ $\int_a^b [f(x)+c]\,v(x)\,dx = 0$, $\forall v \in C(a,b)$ with $\int_a^b v(x)\,dx = 0$

In particular, take $c = -\frac{1}{b-a}\int_a^b f(x)\,dx$, so that $\int_a^b f+c = 0$.

Thus, we can test $(**)$ against $v := f+c$ to get $\int_a^b (f+c)^2 = 0$
$\Rightarrow$ $f \equiv -c$. $\square$

Following a similar reasoning to the one in REMARK 3.6, and arguments similar to the ones contained in the above proof, we can obtain a generalized version of the DBR Lemma (which we state without proof).

LEMMA 3.11 (Generalized DBR)

Consider the space

$$V = \{ v \in C(a,b) \mid \int_a^b v(x)\,dx = 0 \}$$

Assume that $F \subseteq V$ is such that $\overline{\operatorname{span} F} = V$, where $\overline{\operatorname{span} V}$ is

$$\overline{\operatorname{span} V} := \{ v \in C(a,b) \mid \exists \{v_n\} \subseteq \operatorname{span} V, \text{ with } \sup_n \|v_n\|_\infty < +\infty \text{ and } v_n \to v \text{ uniformly on each compact } K \subset (a,b) \setminus I \}$$

with $I := \{x_1, .., x_N\}$ is a fixed finite collection of points. Let $f \in C(a,b)$. If

$$\int_a^b f(x)\,v(x)\,dx = 0, \quad \forall v \in F$$

then $f \equiv c$ for some $c \in \mathbb{R}$.

BOUNDARY CONDITIONS (By Examples)

EXAMPLE 1 (DIRICHLET BOUNDARY CONDITIONS)

$$F(u) = \int_0^1 \dot{u}^2 + u^2 \, dx \quad \text{with} \quad u \in X,$$

$$X := \{ u \in C^1[0,1] \mid u(0) = \alpha, u(1) = \beta \}$$

We want to find solutions to

$$\min_{u \in X} F(u).$$

Let us start by computing the first variation. Thus let

$$V = \{ v \in C^1[0,1] \mid v(0) = v(1) = 0 \}$$

So that X is an affine space over V. For $u \in X$, $v \in V$ we get

$$F(u + tv) = \int_0^1 (\dot{u} + t\dot{v})^2 + (u + tv)^2 \, dx =$$

$$= \int_0^1 \dot{u}^2 + 2t \int_0^1 \dot{u}\dot{v} + t^2 \int_0^1 \dot{v}^2 +$$

$$\int_0^1 u^2 + 2t \int_0^1 uv + t^2 \int_0^1 v^2$$

$$= F(u) + t^2 F(v) + 2t \int_0^1 (uv + \dot{u}\dot{v}) \, dx$$

Therefore

$$\delta F(u,v) = \lim_{t\to 0} \frac{F(u+tv) - F(u)}{t} =$$

$$= \lim_{t\to 0} tF(v) + 2\int_0^1 (uv + \dot{u}\dot{v})\,dx$$

$$= 2\int_0^1 (uv + \dot{u}\dot{v})\,dx$$

Therefore the EULER-LAGRANGE EQUATION reads

(*) $$\boxed{\int_0^1 (uv + \dot{u}\dot{v})\,dx = 0, \quad \forall v \in V}$$

Assuming that $u \in C^2[0,1]$, we can integrate (*) by parts to obtain

(**) $$\boxed{\int_0^1 (-\ddot{u} + u)\,v\,dx = 0, \quad \forall v \in V}$$

where we used $v(0) = v(1) = 0$.

<u>NOTATION</u>
- (*) is called 1st INTEGRAL FORM OF (ELE)
- (**) is called 2nd INTEGRAL FORM OF (ELE)

Thus, if u is minimum of F and $u \in C^2[0,1]$, then u solves (**).
As $C_c^\infty(0,1) \subseteq V$, we can apply FLCV (LEMMA 3.4) to (**) and obtain

$$-\ddot{u} + u \equiv 0$$

Recalling that u satisfies BC, we then need to solve the
ORDINARY DIFFERENTIAL EQUATION (ODE)

$$\begin{cases} \ddot{u}(x) = u(x) , & \forall x \in (0,1) \\ u(0) = \alpha \\ u(1) = \beta \end{cases} \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{DIRICHLET BOUNDARY CONDITIONS (DBC)}$$

Now this is solved by

$$(*) \qquad u(x) = A\cosh(x) + B\sinh(x)$$

for appropriate A, B (as well known from basic analysis courses).

WARNING Recall that this just proves that if $u \in C^2[0,1]$ is a minimizer for F in X, then u is of the form $(*)$. Showing that u as in $(*)$ is actually a minimum requires a proof (energy estimates)

EXAMPLE 2 (DBC and NEUMANN BOUNDARY CONDITION (NBC))

Same functional F from the previous example, but defined on

$$X = \{ u \in C^1[0,1] \mid u(0) = \alpha \}$$

Note: we do not assign a condition for $u(1)$.

Let us compute the first variation. This time the reference vector space is

$$V = \{ v \in C^1[0,1] \mid v(0) = 0 \}.$$

Note that, as a consequence of the def. of X, we do not need to assign conditions on $v(1)$.

As before, the first variation at $u \in X$ along the direction $v \in V$ is

$$\delta F(u,v) = 2\int_0^1 (uv + \dot{u}\dot{v})\,dx$$

Assuming $u \in C^2[0,1]$ and integrating by parts:

$$\delta F(u,v) = 2\int_0^1 uv\,dx + \underbrace{2\,\dot{u}v\Big|_0^1}_{} - 2\int_0^1 \ddot{u}v\,dx$$

This time this term is not zero, but it is equal to $2\,\dot{u}(1)\,v(1)$

Thus the 2nd integral form of (ELE) is

(ELE) $$\boxed{\int_0^1 (-\ddot{u} + u)\,v\,dx + \dot{u}(1)\,v(1) = 0 \,, \quad \forall v \in V}$$

Thus if $u \in C^2[0,1]$ and u minimizes F in X, then (ELE) holds.
How do we proceed? We cannot apply FLCV or DBR straightforwardly.
So we proceed in 2 steps:

- Step 1 : Consider only test function $v \in V$ such that $v(1)=0$.
  In this case (ELE) reads

$$\int_0^1 (-\ddot{u} + u)\,v\,dx = 0, \quad \forall v \in C^1[0,1] \text{ s.t. } v(0)=v(1)=0$$

In particular (as in EXAMPLE 1) we can apply FLCV to get

$$-\ddot{u} + u \equiv 0$$

and hence the ODE

$$\begin{cases} \ddot{u}(x) = u(x), & \forall x \in (0,1) \\ u(0) = \alpha \end{cases}$$

- Step 2: Now we know that $\ddot{u} \equiv u$. Therefore (ELE) becomes

$$\dot{u}(1)\,v(1) = 0, \quad \forall v \in V$$

Thus, by testing against $v \in V$ s.t. $v(1) \neq 0$ we get

$$\dot{u}(1) = 0$$

In total, we found that u solves

$$\begin{cases} \ddot{u}(x) = u(x), & \forall x \in (0,1) \\ u(0) = \alpha & \text{(DIRICHLET BOUNDARY CONDITION)} \\ \dot{u}(1) = 0 & \text{(NEUMANN BOUNDRY CONDITION NBC)} \end{cases}$$

NOTICE: By not imposing a DIRICHLET BOUNDARY CONDITION on $u(1)$ for $u \in X$, we see that minimizers must satisfy a homogeneous condition on $\dot{u}(1)$.

This will be true in general. Also note that the NBC is of one less order than the highest derivative appearing in F.

EXAMPLE 3 (NEUMANN BOUNDARY CONDITIONS - NBC)

F as before but $X := C^1[0,1]$, with no additional conditions.

Note that in this case it is trivially true that $u \equiv 0$ minimizes F. However, for instructive purposes, let us ignore this fact and proceed with our usual method.

This time the ref. vector space is $V = C^1[0,1]$. The first variation is always the same,

$$\delta F(u,v) = 2\int_0^1 (uv + \dot{u}\dot{v})\,dx.$$

Assuming that $u \in C^2[0,1]$ minimizes F on X, and integrating by parts

(ELE) $$\boxed{\int_0^1 (-\ddot{u} + u)\, v\, dx + \dot{u}(1)v(1) - \dot{u}(0)v(0) = 0, \quad \forall v \in V}$$

We now proceed in 2 Steps:

- Step 1: Test (ELE) against $v \in C_c^\infty(0,1) \subseteq V$, so that

$$\int_0^1 (-\ddot{u} + u)\, v\, dx = 0, \quad \forall v \in C_c^\infty(0,1)$$

Thus FLCV implies

$$-\ddot{u} + u \equiv 0$$

- Step 2: Since $-\ddot{u}+u\equiv 0$, (ELE) becomes

$$(*) \quad \dot{u}(1)v(1)-\dot{u}(0)v(0)=0, \qquad \forall v\in V$$

Testing $(*)$ against $v\in V$ s.t. $v(0)\neq 0$, $v(1)=0$ yields

$$\dot{u}(0)=0$$

Testing $(*)$ against $v\in V$ s.t. $v(0)=0$, $v(1)\neq 0$ yields

$$\dot{u}(1)=0$$

In total, u solves

$$\begin{cases} \ddot{u}(x)=u(x), & x\in(0,1) \\ \dot{u}(0)=0 \\ \dot{u}(1)=0 \end{cases} \quad \left.\right\} \text{NEUMANN BOUNDARY CONDITIONS (NBC)}$$

EXAMPLE 4 (PERIODIC BOUNDARY CONDITIONS – PBC)

F as before, but

$$X=\{u\in C^1[0,1] \mid u(0)=u(1)\}$$

(Also now the solution is trivially $u\equiv 0$. But let's ignore this).

Note X is vector space, so we can take $V=X$. The first variation δF is the same. Assuming $u\in C^2[0,1]$ minimizes F on X and integrating by parts:

(ELE) $\boxed{\int_0^1 (-\ddot{u} + u) v \, dx + v(0)\{\dot{u}(1) - \dot{u}(0)\} = 0, \quad \forall v \in V}$

where we used that $v(0) = v(1)$. We proceed in 2 steps:

- Step 1 As usual, we can test against all $v \in C_c^\infty(0,1) \subseteq V$ and get

$$-\ddot{u} + u \equiv 0$$

- Step 2: We know that

$$v(0)\{\dot{u}(1) - \dot{u}(0)\} = 0, \quad \forall v \in V$$

Testing against $v \in V$ with $v(0) \neq 0$ (and $v(1) = v(0)$) we conclude

$$\dot{u}(0) = \dot{u}(1)$$

Recalling that $u(0) = u(1)$ as $u \in X$, we then get

$$\begin{cases} \ddot{u}(x) = u(x), & \forall x \in (0,1) \\ u(0) = u(1) \\ \dot{u}(0) = \dot{u}(1) \end{cases}$$

PERIODIC BOUNDARY CONDITIONS (PBC)

EXAMPLES F the same, $X = \{u \in C^1[0,1] \mid u(1) = u(0) + 2\}$

X is not a vector space. It is however affine space over

$$V = \{C^1[0,1] \mid v(0) = v(1)\}$$

By very similar calculations to the previous 4 examples, we get that if $u \in C^2[0,1]$ minimizes F over X, then

$$\begin{cases} \ddot{u}(x) = u(x), & \forall x \in (0,1) \\ u(1) = u(0) + 5 & \text{(This was enforced in } X\text{)} \\ \dot{u}(0) = \dot{u}(1) & \text{(NBC / PBC)} \end{cases}$$

EXAMPLE 6 (TOO MANY BOUNDARY CONDITIONS!)

F the same,

$$X = \{ u \in C^1[0,1] \mid u(\tfrac{1}{2}) = \alpha \}.$$

X is affine over $V = \{ v \in C^1[0,1] \mid v(\tfrac{1}{2}) = 0 \}$. If $u \in C^2[0,1]$ minimizes F over X, we integrate by parts to find

(ELE) $$\boxed{\int_0^1 (-\ddot{u} + u) v \, dx + \dot{u}(1) v(1) - \dot{u}(0) v(0) = 0, \quad \forall v \in V}$$

- Step 1: Define

$$W := \{ v \in C^1[0,1] \mid v(0) = v(\tfrac{1}{2}) = v(1) = 0 \} \subseteq V$$

By (ELE) we have

$$(*) \quad \int_0^1 (-\ddot{u} + u) v \, dx = 0, \quad \forall v \in W$$

Now notice that $\overline{\operatorname{span} W} = C(0,1)$, where the closure is taken WRT the uniform convergence on compact subsets of $[0,1] \setminus \{\frac{1}{2}\}$. Then we can apply the GENERALIZED FLCV (LEMMA 3.7) to (*) and infer

$$\begin{cases} -\ddot{u} + u \equiv 0 \\ u(\frac{1}{2}) = \alpha \quad \text{(this is from } u \in X) \end{cases}$$

• Step 2: As $-\ddot{u} + u \equiv 0$, from (ELE) we get

$$\dot{u}(1)v(1) - \dot{u}(0)v(0) = 0 \quad , \quad \forall v \in V$$

Now just take $v \in V$ s.t. $v(1) = 0$, $v(0) \neq 0$ and $\tilde{v} \in V$ s.t. $\tilde{v}(1) \neq 0$, $\tilde{v}(0) = 0$ and obtain

$$\dot{u}(1) = \dot{u}(0) = 0 .$$

In total, u solves

$$\text{(ODE)} \quad \begin{cases} \ddot{u}(x) = u(x) & , \; x \in (0,1) \\ u(1/2) = \alpha \\ \dot{u}(0) = \dot{u}(1) = 0 \end{cases}$$

As the ODE is of order 2 and we get 3 pointwise conditions, it is very unlikely that (ODE) admits a solution.

Notice that solvin (ODE) is equivalent to solving 2 separate ODEs and then hoping that the solutions can be glued at $1/2$ in a C^2 way where the two ODEs are

$$\text{(P1)} \begin{cases} \ddot{u} = u & \text{in } (0, 1/2) \\ \dot{u}(0) = 0 \\ u(1/2) = \alpha \end{cases} \quad , \quad \text{(P2)} \begin{cases} \ddot{u} = u & \text{in } (1/2, 1) \\ \dot{u}(1) = 0 \\ u(\frac{1}{2}) = \alpha \end{cases}$$

So there are two possibilities:

1) (ODE) admits a solution u $\Rightarrow$ with energy arguments we show that u minimizes F over X.

2) (ODE) does not admit a solution. Thus

$$\min_{u \in X} F(u)$$

admits no minimizer

$$\Downarrow$$

We solve (P1) and (P2), say with solutions $u_1 \in C^1[0,1/2]$, $u_2 \in C^1[1/2,1]$ respectively. Then

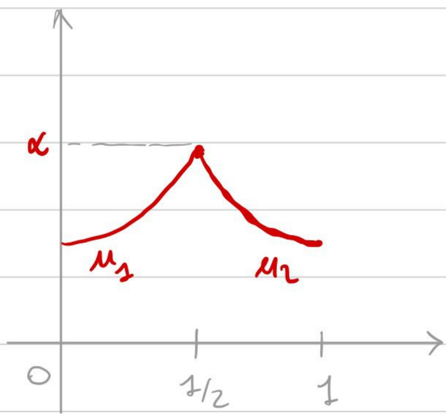
$$\hat{u}(x) := \begin{cases} u_1(x) & \text{if } x \in [0,1/2] \\ u_2(x) & \text{if } x \in [1/2,1] \end{cases}$$

DOES NOT BELONG to $C^1[0,1]$ (otherwise it would be a minimum).

One can show that

$$\inf_{u \in X} F(u) = F(\hat{u}) \leftarrow$$

(Note $F(\hat{u})$ is well defined by splitting the integral)



Idea of the proof:

① Show that $F(u) \geq F(\hat{u})$ for all $u \in X$, by the usual energy estimates

② Construct $\{u_n\} \subseteq X$ s.t. $u_n \to u$ uniformly on each $K \subseteq [0,1] \setminus \{1/2\}$ compact and

$$F(u_n) \to F(\hat{u})$$

This is done in the usual way: ROUNDING the corner of $\hat{u}$ at $x = 1/2$.