LESSON 3 - 17 MARCH 2021

HIGHER ORDER DERIVATIVES

Let X, Y be normed spaces, $U \subseteq X$ open, $F: U \to Y$. Suppose that F is Fréchet diff. in U. Then we can define the map

$$F' : U \longrightarrow \mathcal{L}(X, Y)$$
$$u \longmapsto F'(u)$$

Now we can try and differentiate the above expression. This would yield a second derivative.

DEFINITION 1.10 Assume that F is diff. in U. We say that F is twice FRÉCHET diff. at $u_0 \in U$ if $F' : U \to \mathcal{L}(X, Y)$ is FRÉCHET diff. at u_0, i.e., if $\exists\, A_{u_0} \in \mathcal{L}(X, \mathcal{L}(X, Y))$ s.t.

$$\lim_{\|v\|_X \to 0} \frac{\| F'(u_0+v) - F'(u_0) - A_{u_0}(v) \|_{\mathcal{L}(X,Y)}}{\|v\|_X} = 0$$

NOTATION The second Fréchet derivative is unique, if it exists. We denote it by $F''(u_0) := A_{u_0} \in \mathcal{L}(X, \mathcal{L}(X, Y))$

REMARK 1.11 Introduce the set

$$\mathcal{L}_2(X, Y) := \{ T : X \times X \to Y \mid T \text{ is bilinear, continuous} \}$$

where by continuous we mean $\exists\, M > 0$ s.t.

$$\| T(u, v) \|_Y \leq M \|u\|_X \|v\|_X \,, \quad \forall u, v \in X$$

Then one can show that

$$\mathcal{L}(X, \mathcal{L}(X, Y)) \cong \mathcal{L}_2(X, Y)$$

topologically and as sets (simple exercise)

Therefore $\mathcal{L}_2(X,Y)$ is naturally a normed space (and Banach if Y is Banach) with the norm

$$\|T\|_{\mathcal{L}_2(X,Y)} := \sup_{\|u\|_X, \|v\|_X \leq 1} \|T(u,v)\|_Y$$

DEFINITION 1.12 Let $F: U \to Y$ and assume that F is twice Fréchet diff. at each point of U. Thus we can define

$$F'': U \to \mathcal{L}_2(X,Y)$$
$$u \mapsto F''(u)$$

If F'' is continuous we say that $F \in C^2(U,Y)$

WRT norm on X and operator norm on $\mathcal{L}_2(X,Y)$

In words we say that F is twice continuously diff. in U

THEOREM 1.13

Assume $F: U \to Y$ is twice Fréchet diff at $u_0 \in U$. In particular we have $F''(u_0) \in \mathcal{L}_2(X,Y)$ bilinear and continuous. Then $F''(u_0)$ is also SYMMETRIC, i.e.,

$$F''(u_0)(v,w) = F''(u_0)(w,v) \quad , \forall\, v,w \in X$$

COMMENT ON THE PROOF The proof of THM 1.13 is quite long, and I decided to skip it. The interested reader can find it in the book of HENRI CARTAN – "CALCUL DIFFERENTIEL", 1967 (IN FRENCH) in Theorem 5.1.1 at page 65.

The main ideas of the proof are the following. Introduce the map

$$A(v,w) := \left[F(u_0+v+w) - F(u_0+v) - F(u_0+w) + F(u_0)\right] \in Y$$

for $v, w \in X$ with $\|v\|_X, \|w\|_X$ sufficiently small, so that $u_0+v+w \in U$ and A is well defined.

Notice that A is symmetric. One can show that

$$(*) \quad \| A(v,w) - F''(u_0)(v,w) \|_Y = o\left(\left(\|v\|_X + \|w\|_X \right)^2 \right)$$

(The above is not difficult to show, but it would require further analysis which is outside the scope of this course)

As A is symmetric we can swap v and w in $(*)$ to obtain

$$\| A(v,w) - F''(u_0)(w,v) \|_Y = o\left(\left(\|v\|_X + \|w\|_X \right)^2 \right)$$

Therefore by triangle ineq. and the above estimates we get

$$\| F''(u_0)(v,w) - F''(u_0)(w,v) \|_Y = o\left(\left(\|v\|_X + \|w\|_X \right)^2 \right)$$

Thus $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\| F''(u_0)(v,w) - F''(u_0)(w,v) \|_Y \leq \varepsilon \left(\|v\|_X + \|w\|_X \right)^2, \text{ if } \left(\|v\|_X + \|w\|_X \right) < \delta$$

If $v,w \in X$ are arbitrary, we can find $\lambda \neq 0$ s.t. $\|\lambda v\|_X + \|\lambda w\|_X < \delta$. Applying the above ineq. to $\lambda v, \lambda w$ yields

$$\lambda^2 \| F''(u_0)(v,w) - F''(u_0)(w,v) \|_Y \leq \lambda^2 \varepsilon \left(\|v\|_X + \|w\|_X \right)^2, \quad \forall v,w \in X$$

As $\lambda \neq 0$ and ε is arbitrary, we conclude. $\square$

NOTE THEOREM 1.13 is a generalization of the classical SCHWARZ THEOREM on second derivatives of maps $F : \mathbb{R}^2 \to \mathbb{R}$. Indeed, if F is $C^2(\mathbb{R}^2)$ then

$$F'(u) = \nabla F(u) \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$$

where the application is given by

$$F'(u)(v) = \nabla F(u) \cdot v \,, \qquad \nabla F(u) = \left(\frac{\partial F}{\partial x_1}(u), \frac{\partial F}{\partial x_2}(u) \right)$$

Thus $F' : \mathbb{R}^2 \to \mathcal{L}(\mathbb{R}^2, \mathbb{R})$. Then $F''(u) \in \mathcal{L}(\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2, \mathbb{R})) = \mathcal{L}_2(\mathbb{R}^2, \mathbb{R})$ with application given by

$$F''(u)(v,w) = v^T \nabla^2 F(u)\, w \,, \qquad \nabla^2 F(u) = \begin{pmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2^2} \end{pmatrix}$$

Therefore $F''(u)$ is symmetric $\iff$ $\dfrac{\partial^2 F}{\partial x_1 \partial x_2} = \dfrac{\partial^2 F}{\partial x_2 \partial x_1}$, which is true by the classical Schwarz Theorem.

MORE DERIVATIVES! If $F: U \to Y$ is twice diff. in U then we can define

$$F'': U \longrightarrow \mathcal{L}(X, \mathcal{L}(X,Y))$$
$$u \longmapsto F''(u)$$

The function F'' can be in turn differentiated again, with

$$F'''(u) \in \mathcal{L}(X, \mathcal{L}(X, \mathcal{L}(X,Y)))$$

This procedure can be of course iterated, as F''' defines

$$F''': U \longrightarrow \mathcal{L}(X, \mathcal{L}(X, \mathcal{L}(X,Y)))$$

In general we have that

$$\underbrace{\mathcal{L}(X, \mathcal{L}(X, \mathcal{L}(X, \ldots\ldots \mathcal{L}(X,Y))))}_{n \text{ times}} \cong \mathcal{L}_n(X,Y)$$

where

$$\mathcal{L}_n(X,Y) := \{ T : \overbrace{X \times \cdots \times X}^{n \text{ times}} \to Y \mid T \text{ n-linear, bounded} \}$$

meaning that $\exists M>0$ s.t. $\|T(u_1, \ldots, u_n)\|_Y \le M \|u_1\|_X \cdots \|u_n\|_X$.
The space $\mathcal{L}_n(X,Y)$ is normed by

$$\|T\|_{\mathcal{L}_n(X,Y)} := \sup_{\|u_i\|_X \le 1} \|T(u_1, \ldots, u_n)\|_Y$$

and $\mathcal{L}_n(X,Y)$ is Banach if Y is Banach. In particular the n-th Fréchet derivative is s.t.

$$F^{(n)}(u) \in \mathcal{L}_n(X,Y)$$

THEOREM 1.14 (TAYLOR FORMULA) (For a proof see book by CARTAN page 75)

X, Y Banach, $U \subseteq X$ open, $F: U \to Y$ $(n-1)$-times diff in U and n-times diff at $u_0 \in U$. Then

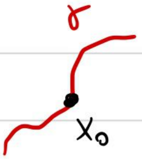
$$F(u_0+v) = F(u_0) + F'(u_0)(v) + \frac{1}{2} F''(u_0)(v,v) + \cdots + \frac{1}{n!} F^{(n)}(u_0)(\underbrace{v, \ldots, v}_{n\text{-times}}) + o(\|v\|_X^n)$$

where $\dfrac{o(\|v\|_X^n)}{\|v\|_X^n} \to 0$ as $\|v\|_X \to 0$.

2. FIRST VARIATION (INDIRECT METHOD)

IDEA The FIRST VARIATION represents the DERIVATIVE of a functional $F: X \to \mathbb{R}$ with X = set

DEFINITION 2.1 Let $x_0 \in X$. A VARIATION at x_0 is a curve $\gamma: [-\delta, \delta] \to X$ s.t. $\gamma(0) = x_0$ (for some $\delta > 0$).



Note this is a scalar function of real variable

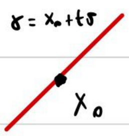
Consider the composition $\psi: [-\delta, \delta] \to \mathbb{R}$, $\psi(t) := F(\gamma(t))$.
If ψ is diff. at $t=0$ we denote

$$\boxed{\delta F(x_0, \gamma) := \psi'(0)}$$

and call $\delta F(\gamma, x_0)$ the FIRST VARIATION of F at x_0 along γ.

NOTE We are not assuming any regularity on F or structure on X

EXAMPLE 2.2 1) $X = \mathbb{R}^d$, $x_0 \in \mathbb{R}^d$ fixed and $F: \mathbb{R}^d \to \mathbb{R}$ differentiable.
For $v \in \mathbb{R}^d$ consider the variation $\gamma(t) = x_0 + tv$, s.t. $\gamma(0) = x_0$.



Then

$$\delta F(x_0, \gamma) := \frac{d}{dt} F(x_0 + tv) \Big|_{t=0} = \nabla F(x_0) \cdot v$$

is the directional derivative of F at x_0 in direction v

2) X normed space, $x_0 \in X$, $F: X \to \mathbb{R}$ Gâteaux diff. at x_0 in every direction. Let $v \in X$ and consider the variation $\gamma(t) := x_0 + tv$.
Then

$$\delta F(x_0, \gamma) := (F \circ \gamma)'(0) = \lim_{t \to 0} \frac{F(\gamma(t)) - F(\gamma(0))}{t} =$$

$$= \lim_{t \to 0} \frac{F(x_0 + tv) - F(x_0)}{t} = F'_g(x_0)(v),$$

The first variation is just the gâteaux derivative of F at x_0 in direction v.

PROPOSITION 2.3 X = Set, $F : X \to \mathbb{R}$. Assume that $x_0 \in X$ is a minimizer for F, that is,

$$F(x) \geq F(x_0) \quad , \forall x \in X$$

Let $\gamma : [-\delta, \delta] \to X$ s.t. $\gamma(0) = x_0$. If $\psi = F \circ \gamma$ is differentiable at $t=0$ then

$$\boxed{\delta F(x_0, \gamma) = 0 .}$$

Proof By assumption ψ is diff at $t=0$. Therefore

$$\psi'(0) = \lim_{t \to 0} \frac{\psi(t) - \psi(0)}{t} = \lim_{t \to 0^+} \frac{\psi(\gamma(t)) - \psi(\gamma(0))}{t} = \lim_{t \to 0^+} \frac{F(\gamma(t)) - F(x_0)}{t} \geq 0$$

So that $\delta F(x_0, \gamma) \geq 0$. Similarly:

$$\psi'(0) = \lim_{t \to 0^-} \frac{\psi(t) - \psi(0)}{t} = \lim_{t \to 0^-} \frac{F(\gamma(t)) - F(x_0)}{t} \leq 0$$

So $\delta F(x_0, \gamma) \leq 0 \quad \Rightarrow \quad \delta F(x_0, \gamma) = 0$. $\square$

THE CASE OF AFFINE SPACES

REMINDER Let V real vector space. A set X is called an AFFINE SPACE with REFERENCE SPACE V if it is defined the addition operation

$$+ : X \times V \to X$$

with properties:

1) $x + 0 = x$, $\forall x \in X$, where $0 \in V$ is the zero in V
2) $x + (v + w) = (x + v) + w$, $\forall x \in X$, $\forall v, w \in V$
3) For every $x \in X$ the map $V \to X$, $v \mapsto x + v$ is a bijection

From the above reminder, the only important part for us is that:
If X is affine space with ref vector space V, then

$$\boxed{x \in X,\ v \in V \ \Rightarrow\ x + v \in X}$$

DEFINITION 2.4 X affine space with reference V. Let $x_0 \in X$. A **VARIATION** at x_0 in direction $v \in V$ is a curve $\gamma(t) := x_0 + tv \in X$

(Thus we restrict ourselves to the case of straight line variations)

REMARK 2.5 Notice that $\gamma : \mathbb{R} \to X$ by def. of affine space (i.e. γ is defined for all $t \in \mathbb{R}$). If $\psi := F \circ \gamma$ is diff at $t=0$ then

$$\psi'(0) = \lim_{t \to 0} \frac{\psi(t) - \psi(0)}{t} = \lim_{t \to 0} \frac{F(x_0 + tv) - F(x_0)}{t}.$$

Therefore the **FIRST VARIATION** reads

$$\boxed{\delta F(x_0, v) := \delta F(x_0, \gamma) = \lim_{t \to 0} \frac{F(x_0 + tv) - F(x_0)}{t}}$$

DEFINITION 2.6 X affine space over V, $F : X \to \mathbb{R}$. If $\delta F(v, x_0)$ exists then

$$\boxed{\delta F(x_0, v) = 0}$$

is called **EULER–LAGRANGE EQUATION (ELE)**. If $x_0 \in X$ is such that (ELE) holds for all $v \in V$ then x_0 is called a **CRITICAL POINT OF F** (or **STATIONARY** POINT)

REMARK 2.7 X affine space over V, $F : X \to \mathbb{R}$ s.t. $\delta F(x_0, v)$ exists $\forall v \in V$. If x_0 minimizes F then x_0 is a CRITICAL POINT, i.e.

$$\delta F(x_0, v) = 0 \quad , \quad \forall v \in V$$

Proof Apply PROPOSITION 2.3 to $\gamma(t) := x_0 + tv$. $\square$

THREE EXAMPLES

Let $a<b$, and $A<B$. Consider the set

$$X = \{ u \in C^1[a,b] \mid u(a) = A , \; u(b) = B \}$$

Then X is an affine space with reference vector space

$$V = \{ u \in C^1[a,b] \mid u(a) = 0 , \; u(b) = 0 \}$$

We consider functionals in integral form:

$$u \in X \longmapsto \int_a^b L(x, u(x), u'(x)) \, dx$$

for LAGRANGIANS $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Specifically, consider $F, G, H : X \to \mathbb{R}$ defined by

$$F(u) := \int_a^b |\dot{u}(x)|^2 \, dx , \quad G(u) := \int_a^b |u(x)| \, dx , \quad H(u) = \int_a^b |\dot{u}(x)|^{1/2} dx$$

GOAL: We want to solve

$$\min_{u \in X} F(u) , \quad \min_{u \in X} G(u) , \quad \min_{u \in X} H(u)$$

and possibly characterize the solutions (if they exist!)

MINIMIZATION FOR F

By REMARK 2.7 we know that minimizers solve (ELE). Therefore, let us compute the first variation of F.

To do that, we compute the Fréchet derivative of F (This is actually not needed. The Gâteaux derivative of F would be sufficient, as seen in EXAMPLE 2.2 point (2). However we compute the Fréchet derivative as an exercise.)

PROPOSITION 2.8 Set $\tilde{X} := C^1[a,b]$ with norm $\|u\|_{C^1} := \|u\|_\infty + \|\dot{u}\|_\infty$. Extend F by

$$F(u) = \int_a^b |\dot{u}(x)|^2 \, dx, \qquad u \in \tilde{X}$$

Then F is Fréchet differentiable in $\tilde{X}$ with $F'(u) \in \mathcal{L}(\tilde{X}, \mathbb{R})$ given by

$$F'(u)(v) = 2\int_a^b \dot{u}(x)\,\dot{v}(x)\, dx, \qquad \forall v \in \tilde{X}$$

Proof Let us start by computing the Gâteaux derivative of F at $u \in \tilde{X}$ in direction $v \in \tilde{X}$. We have

$$F(u+tv) = \int_a^b |\dot{u} + t\dot{v}|^2 dx = \int_a^b |\dot{u}|^2 dx + 2t\int_a^b \dot{u}\dot{v}\, dx + t^2 \int_a^b |\dot{v}|^2 dx$$

Therefore

$$F_g'(u)(v) = \lim_{t \to 0} \frac{F(u+tv) - F(u)}{t} =$$

$$= 2\int_a^b \dot{u}\dot{v}\, dx + \lim_{t\to 0} t \int_a^b |\dot{v}|^2 dx$$

$$= 2\int_a^b \dot{u}\dot{v}\, dx$$

Now notice that $F_g'(u) \in \mathcal{L}(\tilde{X}, \mathbb{R})$: In fact

$$|F_g'(u)(v)| \leq 2\int_a^b |\dot{u}|\,|\dot{v}|\, dx \leq 2\|v\|_{C^1} \int_a^b |\dot{u}|\, dx$$

so that

$$\sup_{\|v\|_{C^1} \leq 1} |F_g'(u)(v)| \leq 2\int_a^b |\dot{u}|\, dx < +\infty$$

and so $F_g'(u) \in \mathcal{L}(\tilde{X}, \mathbb{R})$. Note that this is true for all $u \in \tilde{X}$. Consider

$$F_g' : \tilde{X} \longrightarrow \mathcal{L}(\tilde{X}, \mathbb{R})$$
$$u \longmapsto F_g'(u)$$

Then F'_g is continuous in $\tilde{X}$. Indeed, let $u, v \in \tilde{X}$:

$$\| F'_g(u) - F'_g(v) \|_{\mathcal{L}(\tilde{X}, \mathbb{R})} = \sup_{\|w\|_{C^1} \leq 1} | F'_g(u)(w) - F'_g(v)(w) |$$

$$\leq 2 \sup_{\|w\|_{C^1} \leq 1} \int_a^b |\dot{u} - \dot{v}| \, |\dot{w}| \, dx$$

$$\leq 2 \int_a^b |\dot{u} - \dot{v}| \, dx \leq 2(b-a) \|u - v\|_{C^1}$$

showing continuity. We can then apply THEOREM 1.9 and conclude that F is Fréchet diff. in $\tilde{X}$ with $F'(u) = F'_g(u)$. $\square$

Since X affine space, REMARK 2.5 tells us that for $u_0 \in X$ we have

$$\delta F(u_0, v) = \lim_{t \to 0} \frac{F(u_0 + tv) - F(u_0)}{t}, \quad \forall v \in V$$

if the limit exists. Note that $X, V \subseteq \tilde{X}$. Therefore, as we just computed the Fréchet derivative of F (PROPOSITION 2.8), we get

$$F'(u_0)(v) = F'_g(u_0)(v) = \lim_{t \to 0} \frac{F(u_0 + tv) - F(u_0)}{t}$$

and so in particular

$$\boxed{\delta F(u_0, v) = 2 \int_a^b \dot{u}(x) \, \dot{v}(x) \, dx, \quad \forall u_0 \in X, v \in V}$$

We now look for solutions to (ELE) in order to find STATIONARY POINTS.

Thus assume $u_0 \in X$ is a min. of F over X. By REMARK 2.7 we have

$$\boxed{\delta F(u_0, v) = 0 \,, \quad \forall v \in V} \qquad \text{(ELE)}$$

Assuming also that $u_0 \in C^2[a,b]$ we get

$$0 = \delta F(u_0, v) = 2 \int_a^b \dot{u}_0 \dot{v} \, dx$$

(Integrate by parts WRT to $(\dot{u} v)' = \ddot{u} v + \dot{u} \dot{v}$)

$$= 2 \left[\dot{u}_0 v \right]_{x=a}^{x=b} - 2 \int_a^b \ddot{u}_0 v \, dx$$

(as $v(a) = v(b) = 0$)

$$= -2 \int_a^b \ddot{u}_0 v \, dx$$

Thus

$$(*) \quad \int_a^b \ddot{u}_0 v \, dx = 0 \,, \quad \forall v \in V$$

It looks like $(*)$ can hold iff $\ddot{u}_0 \equiv 0$. Let's say this is true (it actually is true and we will show it soon). Therefore, as $u_0(a) = A$, $u_0(b) = B$, we have

$$\ddot{u}_0 \equiv 0 \implies u_0 \text{ is straight line connecting } (a, A) \text{ and } (b, B)$$

WARNING This does not prove that u_0 minimizes F over X. We just proved that:

"If $u_0 \in C^2[a,b]$ is a min. of F over $X \Rightarrow u_0$ straight line"

PROPOSITION 2.9 Let $u_0 \in X$ be a straight line. Then u_0 is the unique solution to

$$\min_{u \in X} F(u) \,.$$

Recall: $X = \{ u \in C^1[a,b] \mid u(a) = A, u(b) = B \}$, $F(u) := \int_a^b |\dot{u}|^2 dx$, $A < B$

<u>Proof</u> Let $w \in X$ be arbitrary. We need to prove:

1) $F(u_0) \leq F(w), \ \forall w \in X$ (thus u_0 is a minimizer)

2) $F(u_0) = F(w) \iff u_0 = w$ (thus u_0 is unique minimizer)

Let us show (1): As $u_0, w \in X$ then $v := w - u_0 \in V$, since $v(a) = v(b) = 0$.

$$
\begin{aligned}
F(w) = F(u_0 + v) &= \int_a^b |\dot{u}_0 + \dot{v}|^2 dx \\
&= \int_a^b |\dot{u}_0|^2 \, dx + 2\int_a^b \dot{u}_0 \dot{v} \, dx + \int_a^b |\dot{v}|^2 dx \\
&= F(u_0) + 2\int_a^b \dot{u}_0 \dot{v} \, dx + F(v)
\end{aligned}
$$

Now u_0 is actually $C^2[a,b]$ (being a straight line). As $v(a) = v(b) = 0$, we can proceed as above (integrating by parts to obtain)

$$
\int_a^b \dot{u}_0 \dot{v} \, dx = \underbrace{\Big[\dot{u}_0 v \Big]_a^b}_{=0 \text{ as } v(a)=v(b)=0} - \underbrace{\int_a^b \ddot{u}_0 v \, dx}_{=0 \text{ as } \ddot{u}_0 = 0, \text{ since } u_0 \text{ is a line}} = 0
$$

Thus

$$
F(w) = F(u_0) + F(v) \geq F(u_0), \quad \text{(Since } F \geq 0 \text{ by definition)}
$$

showing (1). Let us prove (2): We know that

$$
F(w) = F(u_0) + F(v)
$$

So $F(w) = F(u_0)$ iff $F(v) = 0$. By def of F this is true iff $\dot{v} \equiv 0$. Thus $v \equiv$ constant. Since $v(a) = v(b) = 0 \implies v \equiv 0$. Recalling that $v = w - u_0$, we infer $w \equiv u_0$, as claimed. $\square$

Recall: $X = \{ u \in C^1[a,b] \mid u(a) = A, u(b) = B \}$

MINIMIZATION FOR G

$$G(u) := \int_a^b |\dot{u}| \, dx, \qquad A < B$$

PROPOSITION 2.10

We have that

$$(*) \qquad \min_{u \in X} G(u) = B - A$$

and the minimum exists. Moreover the only solutions to $(*)$ are the monotonic functions, that is, $u_0 \in X$ with $\dot{u}_0 \geq 0$.

Proof Let $u \in X$ be arbitrary. Then

$$G(u) = \int_a^b |\dot{u}(x)| \, dx \geq \left| \int_a^b \dot{u}(x) \, dx \right| = |u(b) - u(a)| = B - A$$

Hence

$$(LB) \qquad G(u) \geq B - A, \qquad \forall u \in X$$

This lower bound is achieved by u_0 straight line between $(a, A), (b, B)$.
Indeed,

$$G(u_0) = \int_a^b |\dot{u}_0(x)| \, dx = \int_a^b \frac{B-A}{b-a} \, dx = B - A$$

Therefore u_0 solves $(*)$, as (LB) implies

$$(**) \qquad G(u) \geq G(u_0) = B - A, \qquad \forall u \in X.$$

In particular $(*)$ holds.

Assume now that $w_0 \in X$ solves $(*)$. Thus $G(w_0) = B - A$. Since (as above)

$$G(w_0) = \int_a^b |\dot{w}_0(x)| \, dx \geq \left| \int_a^b \dot{w}_0(x) \, dx \right| = B - A,$$

we have $G(w_0) = B - A$ iff

$$\int_a^b |\dot{w}_0| \, dx = \left| \int_a^b \dot{w}_0(x) \, dx \right|.$$

But this is true iff $\mathrm{sign}(\dot{w}_0)$ is constant $\iff$ $\dot{w}_0 \geq 0$ (as $A < B$).

$\square$

Recall: $X = \{ u \in C^1[a,b] \mid u(a)=A,\ u(b)=B \}$

$H(u) := \int_a^b \sqrt{|\dot{u}|}\, dx \quad , \quad A<B$

MINIMIZATION FOR H

PROPOSITION 2.11 We have that

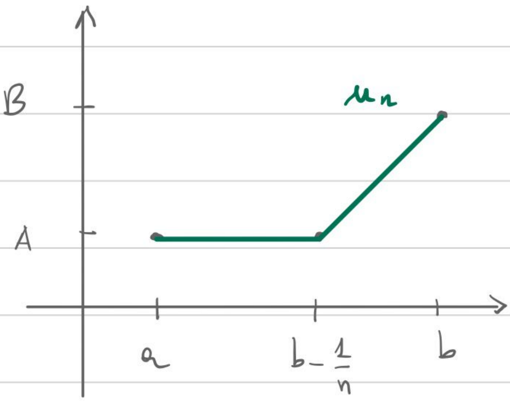
$$(*) \qquad \min_{u \in X} H(u)$$

has no solutions, and

$$(**) \qquad \inf_{u \in X} H(u) = 0 .$$

Proof Let us first show (**). As $H \geq 0$, then $\inf_X H \geq 0$. Thus we need to find $\{u_n\} \subseteq X$ s.t. $H(u_n) \to 0$, so that $\inf H = 0$ follows.

Define $u_n : [a,b] \to \mathbb{R}$ as in the picture below



That is:

- $u_n \equiv A$ in $[a, b-\frac{1}{n}]$
- u_n straight line in $[b-\frac{1}{n}, B]$, so that $u_n(b-\frac{1}{n}) = A$ and $u_n(b) = B$

Then we have

$$H(u_n) = \int_a^b \sqrt{|\dot{u}_n|}\, dx = \int_{b-1/n}^{b} \sqrt{|\dot{u}_n|}\, dx$$

As $\dot{u}_n \equiv 0$ in $[a, b-\frac{1}{n})$ and $\dot{u}_n \equiv \frac{B-A}{b-(b-1/n)} = n(B-A)$ in $(b-1/n, b]$

$$= \int_{b-1/n}^{b} \sqrt{n(B-A)}\, dx =$$

$$= \frac{1}{\sqrt{n}} \sqrt{B-A} \longrightarrow 0 \quad \text{as } n \to +\infty$$

PROBLEM: This almost shows (**). The only issue is that $u_n \notin C^1[a,b]$, as $\dot{u}_n$ has a jump at $x = b - 1/n$.

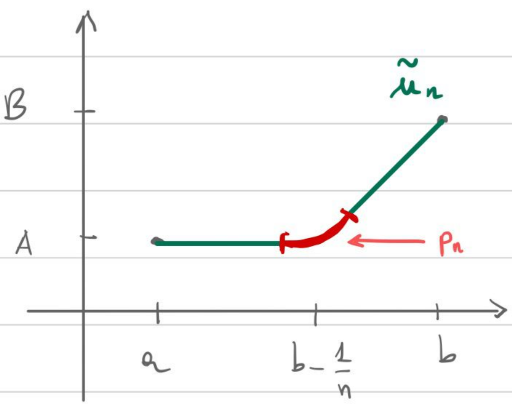
FIX: Smooth u_n around $x = b - 1/n$. For example one could define

$$\tilde{u}_n(x) := \begin{cases} u_n(x) & \text{if } x \in [a, b - \frac{2}{n}] \cup [b - \frac{1}{2n}, b] \\ p_n(x) & \text{if } x \in [b - \frac{2}{n}, b - \frac{1}{2n}] \end{cases}$$

where p_n is a polynomial such that

$$\begin{cases} p_n\left(b - \frac{2}{n}\right) = u_n\left(b - \frac{2}{n}\right), & \dot{p}_n(b - 2/n) = \dot{u}_n(b - 2/n) \\ p_n(b - 1/2n) = u_n(b - 1/2n), & \dot{p}_n(b - 1/2n) = \dot{u}_n(b - 1/2n) \end{cases}$$

so that $\tilde{u}_n \in C^1[a,b]$ and so $\tilde{u}_n \in X$ is admissible. This would look like



Since the region where we changed u_n is infinitesimal as $n \to +\infty$, we get

$$H(\tilde{u}_n) = H(u_n) + o(1) \to 0 \quad \text{as } n \to +\infty$$

(Showing these details will be left as an exercise in the Ex. course)

showing (**). We are left to show that (*) admits no minimizers. Assume by contradiction that the infimum is achieved by some $u_0 \in X$. As (**) holds we get $H(u_0) = 0$, which is possible iff $u_0 \equiv$ constant. However, since $u_0(a) = A$ and $u_0(b) = B$, and since we are assuming $A \neq B$, we get a contradiction.

$\square$

Summary

We considered functionals on $X = \{ u \in C^1(a,b) \mid u(a) = A ,\ u(b) = B \}$,

$$F(u) = \int_a^b \dot{u}^2 \, dx \quad , \qquad G(u) = \int_a^b |u| \, dx \quad , \quad H(u) = \int_a^b \sqrt{|\dot{u}|} \, dx$$

For these the solutions were as follows:

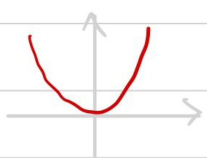
$\min_{u \in X} F(u)$ — UNIQUE MINIMIZER: u_0 STRAIGHT LINE BETWEEN $(a,A),(b,B)$

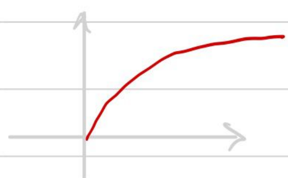
$\min_{u \in X} G(u)$ — INFINITELY MANY MINIMIZERS: ALL THE MONOTONIC FUNCTIONS $\dot{u} \geq 0$

$\min_{u \in X} H(u)$ — NO MINIMIZERS

To understand what is going on, let us considers the LAGRANGIANS associated to F, G, H

$$L_F(x,s,p) = p^2 \quad , \qquad L_G(x,s,p) = |p| \quad , \qquad L_H(x,s,p) = |p|^{1/2}$$





We observe:

- L_F is STRICTLY CONVEX in p $\rightsquigarrow$ $\exists!$ minimizer $u_0 \in C^\infty[a,b]$ (smooth)
- L_G is CONVEX in p, but NOT STRICTLY $\rightsquigarrow$ $\exists$ minimizer, but no uniqueness, no smooth
- L_H is NOT CONVEX in p $\rightsquigarrow$ Non Existence and no regularity