# LESSON 1 - 3 MARCH 2021

**CALCULUS OF VARIATIONS:** The study of minimization problems

$X$ = set, $F: X \to \mathbb{R}$ (often $\mathbb{R} \cup \{\pm\infty\}$) function

We want to solve:

$$\min \{F(u) \mid u \in X\} \quad , \quad \operatorname{argmin} \{F(u) \mid u \in X\}$$

This is a real number, called MINIMUM

These are elements of $X$, called MINIMIZERS

We will look at the following classes of methods to study minimization problems:

- INDIRECT METHODS
- DIRECT METHODS
- RELAXATION
- $\Gamma$ - CONVERGENCE

Let us see some basic examples to see what the above words mean:

**EXAMPLE 1** $F: \mathbb{R} \to \mathbb{R}$, $F(x) = x^2 - 4x$. What is $\min\{F(x) \mid x \in \mathbb{R}\}$?

INDIRECT METHOD: Find candidate minimizers. If $\hat{x}$ is min. then $F'(\hat{x}) = 0$. Now $F'(x) = 2x - 4 = 0$ iff $x = 2$. So $\hat{x} = 2$ is a candidate minimizer and minimum value is $F(2) = -4$

CLAIM $\overset{1}{x}=2$ is the UNIQUE minimizer of $F$. Thus

$$\min_{\mathbb{R}} F = -4 \quad , \quad \operatorname{argmin}_{\mathbb{R}} F = \{2\} \subseteq \mathbb{R}$$

Proof We need to show that

1) $F(x) \geq F(2) \quad \forall x \in \mathbb{R} \qquad (\hat{x}=2$ is minimizer$)$
2) $F(x) > F(2) \quad \forall x \in \mathbb{R} \setminus \{2\} \qquad (\overset{1}{x}=2$ is unique min.$)$

1) $F(x) \geq F(2) \iff x^2-4x \geq -4 \iff (x-2)^2 \geq 0 \iff x=2$
2) $F(x) > F(2) \iff (x-2)^2 > 0 \iff x \neq 2$ $\square$

EXAMPLE 2 DIRECT METHOD: proving existence of a minimizer by general results

Ex: $F: \mathbb{R} \to \mathbb{R}$ continuous and coercive, i.e.,

$$\lim_{|x| \to +\infty} F(x) = +\infty$$

Then $\exists$ minimizer by Weierstrass Theorem

Example 3 RELAXATION: This technique is relevant when a minimizer does not exist, e.g.,

$$\circledast \quad \min\{(x^2-2)^2 \mid x \in \mathbb{Q}\}$$

Solution of $\circledast$ would be $\overset{1}{x} = \pm\sqrt{2}$ which is not in $\mathbb{Q}$. So in this case there is no minimum. But we are left with 2 questions

1) What is

$$\inf \{ (x^2-2)^2 \mid x \in \mathbb{Q} \} \quad ?$$

2) If $\{x_n\}$ is MINIMIZING SEQUENCE, i.e.

$$F(x_n) \to \inf \{ F(x) \mid x \in \mathbb{Q} \}, \quad F(x) = (x^2-2)^2$$

what can we say about accumulation points of $\{x_n\}$ ?

Answer: 1) As we guessed, min over $\mathbb{R}$ would be $\hat{x} = \pm\sqrt{2}$, so

$$\inf \{ (x^2-2)^2 \mid x \in \mathbb{Q} \} = F(\pm\sqrt{2}) = 0$$

2) $x_n \to \sqrt{2}$ OR $x_n \to -\sqrt{2}$ (up to subsequences)

Relaxation is useful to treat problems such as $(*)$. To ensure that a minimizer exists one could, for example,

- Extend $F$ over some set $\hat{X}$ with $\hat{X} \supseteq X$ ($\hat{X} = \mathbb{R}$ for $(*)$)
- Change $F$ so that a minimizer is more likely to exist

**EXAMPLE 4** $\Gamma$-CONVERGENCE: We have a family of problems

$$\min \{ F_n(x) \mid x \in X \}, \quad F_n : X \to \mathbb{R}, \quad n \in \mathbb{N}$$

What happens as $n \to +\infty$ ? We hope to find $F_\infty : X \to \mathbb{R}$ such that

1) $\min \{ F_n \mid x \in X \} \to \min \{ F_\infty \mid x \in \mathbb{R} \}$ as $n \to +\infty$

2) If $x_n \in \operatorname{argmin} \{ F_n \mid x \in X \}$ then $x_n \to x_\infty$ with

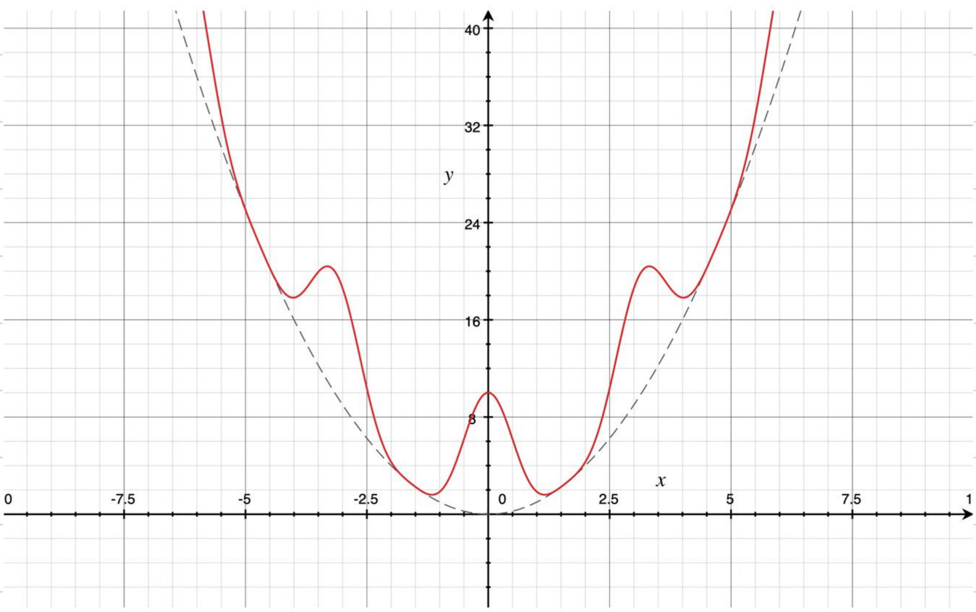
$$x_\infty \in \operatorname{argmin} \{ F_\infty \mid x \in X \}$$

$F_\infty$ is the $\Gamma$-limit of $\{F_n\}$ as $n \to +\infty$

For example consider

$$m_n = \min \{ F_n(x) \mid x \in \mathbb{R} \} , \quad F_n(x) = x^2 + n \cos^4 x$$

What is the limit of $m_n$?




Dashed $y = x^2$

Red $F_n$, $n = 10$

- $F_n$ is sum of two positive terms
- $x^2$ small $\iff x \simeq 0$
- $n\cos^4 x$ small $\iff \cos x \simeq 0$

True when $x = \pm \frac{\pi}{2}$

Indeed one has

1) $m_n \to \left(\frac{\pi}{2}\right)^2$ as $n \to +\infty$

2) $\{x_n\}$ minimizing sequence converges (up to subsequences) to $\pm\pi/2$.

3) The $\Gamma$-limit is

$$F_\infty(x) = \begin{cases} x^2 & \text{if } \cos x = 0 \\ +\infty & \text{otherwise} \end{cases}$$

# INTEGRAL FUNCTIONALS

This course mainly focusses on integral functionals

$X =$ some functions space, e.g.,

$$X = C^k[a,b] = \{ u : [a,b] \to \mathbb{R} \mid u \text{ } k\text{-times continuously differentiable} \}$$

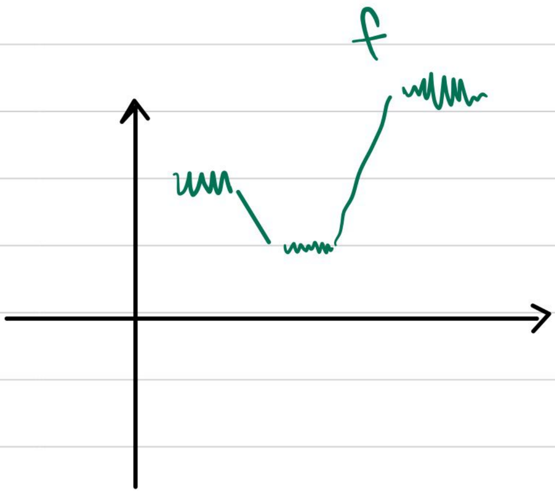
and $F : X \to \mathbb{R}$ is of the form

$$F(u) := \int_a^b L(x, u(x), u'(x), \dots, u^{(k)}(x))\, dx$$

with $L : [a,b] \times \mathbb{R}^k \to \mathbb{R}$ LAGRANGIAN

Typically $L : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L = L(x, s, p)$

**EXAMPLE 1** (DENOISING)



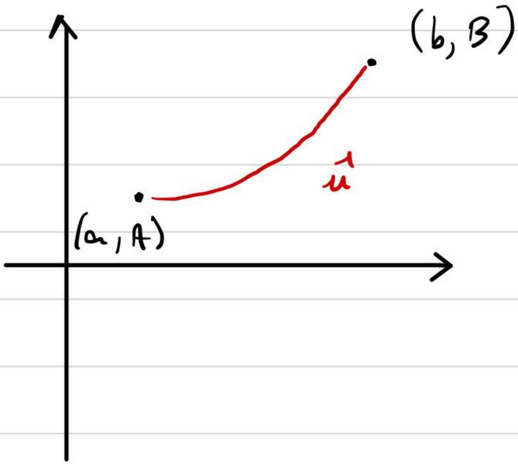

We receive a signal $f : [0,1] \to \mathbb{R}$ which we want to denoise

This task is achieved by solving

$$\hat{u} \in \operatorname{argmin} \left\{ \int_0^1 \dot{u}^2 + (u-f)^2 \, dx \;\middle|\; u \in C^1[0,1] \right\}$$

Note
- $\dot{u}$ penalizes oscillations
- $u - f$ penalizes discrepancy from the noisy signal $f$

**Example 2** (Hanging Rope) Find the profile of a rope hanging at $(a, A)$, $(b, B)$




The energy of a profile $u : [a,b] \to \mathbb{R}$ is modelled by

$$E(u) = \int_a^b \underbrace{\dot{u}^2}_{\text{elastic energy}} + \underbrace{u}_{\text{potential energy}} \, dx \quad , \; u(a) = A, u(b) = B$$

Note 1) $u$ can be negative which lowers $E$

2) Due to boundary conditions, if $u \ll 0$ then $\dot{u} \gg 0 \Rightarrow E$ higher

The solution

$$\hat{u} \in \operatorname{argmin} \{ E(u) \mid u \in C^1[a,b], u(a) = A, u(b) = B \}$$

will be a balance between (1) and (2)

## PROBLEMS WE WILL NOT TALK ABOUT

1) GEOMETRIC FUNCTIONALS:

- DIDO'S PROBLEM: $\min \{ \operatorname{Area}(\partial V) \mid V \subseteq \mathbb{R}^3, \operatorname{Vol}(V) = 1 \}$

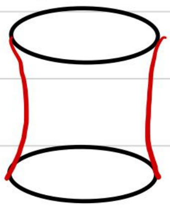
Intuitively the sol is a sphere. However proving it when not requiring regularity on $V$ requires advanced tools (Geometric Measure Theory)

• PLATEAU'S PROBLEM: Given $\gamma$ collection of curves in $\mathbb{R}^3$ find

$$\min \{ \text{Area}(S) \mid S \subseteq \mathbb{R}^3 \text{ surface}, \ \partial S = \gamma \}$$

Again this requires GMT

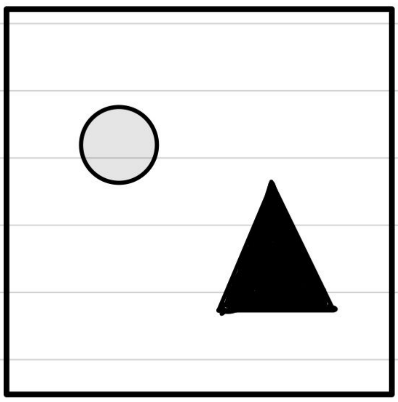
$\gamma$



$\hat{S}$ CATENOID is solution

2) IMAGING FUNCTIONALS: used for tasks such as DEnoising, Segmentation, reconstruction of medical data. Usually

$\Omega$



$$u : \Omega \to \mathbb{R} \quad , \quad \Omega \subseteq \mathbb{R}^2, \mathbb{R}^3$$

$u$ encodes gray-scale value of pixels of a picture in the frame $\Omega$

Example: To segment the image at the left, i.e. find contours of shapes within it, one could minimize the MUMFORD-SHAH functional:

$$F(u, K) = \underbrace{\int_{\Omega \setminus K} |\nabla u|^2 \, dx}_{(I)} + \underbrace{\int_{\Omega} |u - f|^2 \, dx}_{(II)} + \underbrace{\text{Length}(K)}_{(III)}$$

• $f$ is the original picture, generally noisy. Want to clean $f$ and detect edges $K$ within it

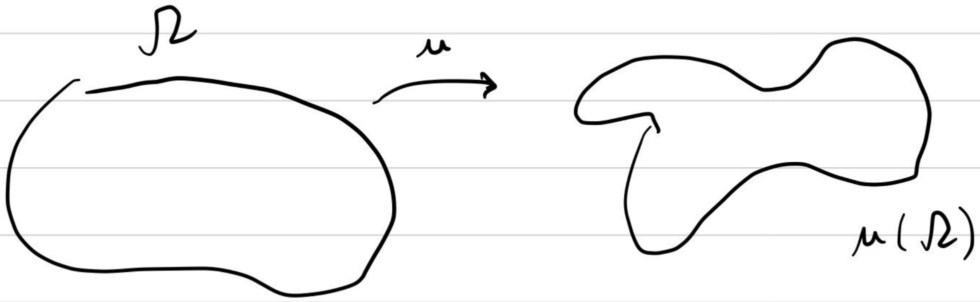
• (I): enforces smoothness of $u$ outside of $K$ (we don't want to pay energy for the natural transitions)

- (II) : Enforces the clean image $u$ to be close to the original $f$
- (III) : Forces short contours

A solution is then

$$(\hat{u}, \hat{K}) \in \operatorname{argmin} \{ F(u,K) \mid K \subseteq \Omega \text{ compact}, \ u \in C^1(\Omega \setminus K) \}$$

3. VECTORIAL PROBLEMS : For example in materials science $\Omega \subseteq \mathbb{R}^3$ represents the reference configuration of an elastic body, $u : \Omega \to \mathbb{R}^3$ is a deformation




The elastic energy of the deformed configuration is

$$E(u) = \int_\Omega W(\nabla u) \, dx \quad , \quad W : \mathbb{R}^{3 \times 3} \to [0, +\infty]$$

In this case the problem of minimizing $E$ is vectorial, and the analysis requires advanced tools (quasi-convexity, etc)

An equilibrium configuration given boundary data $g : \partial\Omega \to \mathbb{R}^3$ is

$$\min \{ E(u) \mid u \in C^1(\Omega; \mathbb{R}^3), \ u = g \text{ on } \partial\Omega \}$$

# BASIC FUNCTIONAL ANALYSIS (Revision)

REFERENCE: J.B. CONWAY "A COURSE IN FUNCTIONAL ANALYSIS" SECOND EDITION, SPRINGER, 1997

**METRIC SPACE** $X$ set, $d: X \times X \to [0, +\infty)$. We say that $d$ is a **METRIC** over $X$ if

- $d(x,y) = 0 \iff x = y$
- $d(x,y) = d(y,x)$, $\forall x,y \in X$ (Symmetric)
- $d(x,y) \leq d(x,z) + d(y,z)$, $\forall x,y,z \in X$ (Triangle inequality)

The pair $(X,d)$ is called a **Metric Space**

**CONVERGENCE** For $\{x_n\} \subseteq X$ we say that $x_n \to x_0$ if

$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $d(x_n, x_0) < \varepsilon$ if $n \geq N_\varepsilon$

**CAUCHY SEQUENCE** $\{x_n\} \subseteq X$ is a Cauchy sequence if

$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $d(x_n, x_m) < \varepsilon$ if $n,m \geq N_\varepsilon$

**COMPLETENESS** A metric space $(X,d)$ is complete if every Cauchy sequence $\{x_n\} \subseteq X$ converges to some $x_0 \in X$.

**Topology generated by $d$** $(X,d)$ metric space. Define

$$\tau := \{ A \subseteq X \mid \forall x \in A, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq A \}$$

with $B_\varepsilon(x) := \{ y \in X \mid d(x,y) < \varepsilon \}$. Then $\tau$ is a **TOPOLOGY** over $X$. The sets in $\tau$ are called OPEN. $A \subseteq X$ is CLOSED if $A^c := X \setminus A$ is open.

**NOTATION** $(X,\tau)$ topological space, $A \subseteq X$. We denote by

- $\mathring{A}$ the **INTERIOR** of $A$: $\mathring{A} = \bigcup \{ O \mid O \subseteq A, O \text{ open} \}$
- $\bar{A}$ the **CLOSURE** of $A$: $\bar{A} = \bigcap \{ C \mid A \subseteq C, C \text{ closed} \}$

In other words:

- $\mathring{A}$ is the largest open set contained in $A$
- $\bar{A}$ is the smallest closed set which contains $A$

**DENSITY** $(X, d)$ metric space. $D \subseteq X$ is DENSE in $X$ if $\bar{D} = X$

**SEPARABILITY** $(X, d)$ metric space is SEPARABLE if $\exists$ a COUNTABLE set $D \subseteq X$ which is dense, i.e., $\bar{D} = X$

**LIMITS** $(X, d_X)$, $(Y, d_Y)$ metric spaces, $U \subseteq X$ open, $F : U \to Y$, $x_0 \in U$. We say that $F(x) \to L$ as $x \to x_0$, in symbols

$$\lim_{x \to x_0} F(x) = L,$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(F(x), L) < \varepsilon$ if $d_X(x, x_0) < \delta$

**CONTINUITY** $(X, d_X)$, $(Y, d_Y)$ metric spaces, $U \subseteq X$ open, $F : U \to Y$. We say that $F$ is continuous at $x_0 \in U$ if $F(x) \to F(x_0)$ as $x \to x_0$. $F$ is continuous in $U$ if it is continuous $\forall x_0 \in U$.

**NORMED SPACE** $X$ vector space over $\mathbb{R}$, $\|\cdot\| : X \to [0, +\infty)$. We say that $\|\cdot\|$ is a norm over $X$ if

- $\|x\| = 0 \iff x = 0$
- $\|\lambda x\| = |\lambda| \, \|x\|$, $\forall \lambda \in \mathbb{R}$, $x \in X$ (1-homogeneous)
- $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$ (subadditive)

The pair $(X, \|\cdot\|)$ is called normed space

**REMARK** $(X, \|\cdot\|)$ normed space. Then $d(x, y) = \|x - y\|$ is a metric over $X$. In particular $X$ is a topological space with $\tau$ induced by $d$. By convention all the topological notions in $X$ are given WRT $\tau$.

**BANACH SPACE** $(X, \|\cdot\|)$ normed space is BANACH if $(X, d)$ with $d(x,y) = \|x-y\|$ is complete.

**LINEAR OPERATORS** $X, Y$ normed spaces, $T: X \to Y$. We say that

- $T$ is LINEAR if $T(\lambda x + y) = \lambda Tx + Ty$, $\forall \lambda \in \mathbb{R}, x, y \in X$
- $T$ is BOUNDED if

$$\sup_{\|x\|_X \leq 1} \|Tx\|_Y < +\infty$$

**FACT** Let $T: X \to Y$ be linear. Then

$$T \text{ is continuous} \iff T \text{ is bounded}$$

**NOTATION** $\mathcal{L}(X,Y) := \{ T: X \to Y \mid T \text{ linear bounded} \}$

$X^* := \mathcal{L}(X, \mathbb{R})$ DUAL SPACE of $X$

**REMARK** 1) $\mathcal{L}(X,Y)$ is a vector space over $\mathbb{R}$, with operations

$$(\alpha T_1 + T_2)(x) := \lambda T_1 x + T_2 x, \quad \forall \alpha \in \mathbb{R}, T_1, T_2 \in \mathcal{L}(X,Y)$$

2) $\mathcal{L}(X,Y)$ is a normed space with norm

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|_X \leq 1} \|Tx\|_Y$$

3) If $Y$ is Banach then $\mathcal{L}(X,Y)$ is Banach

4) $X$ normed space $\Rightarrow$ $X^*$ Banach space

**CONVERGENCES** $(X, \|\cdot\|)$ normed space $\{x_n\} \subseteq X$, $x_0 \in X$, $\{\varphi_n\} \subseteq X^*$ $\varphi_0 \in X^*$,

1) $x_n \to x_0$ **STRONGLY** if $\|x_n - x_0\|_X \to 0$ as $n \to +\infty$
2) $x_n \rightharpoonup x_0$ **WEAKLY** if $\varphi(x_n) \to \varphi(x_0)$, $\forall \varphi \in X^*$
3) $\varphi_n \overset{*}{\rightharpoonup} \varphi_0$ **WEAKLY*** if $\varphi_n(x) \to \varphi_0(x)$, $\forall x \in X$
4) $\varphi_n \to \varphi_0$ **STRONGLY** if $\|\varphi_n - \varphi_0\|_{X^*} \to 0$

**NOTE**

- $x_n \to x_0 \implies x_n \rightharpoonup x_0$
- The reverse implication is not true. For example let

$$X = \ell^p := \left\{ (x_1, x_2, \dots, x_n, \dots) \;\middle|\; \sum_{j=1}^{+\infty} |x_j|^p < +\infty \right\}$$

with $1 < p < +\infty$. Recall that $(X, \|\cdot\|)$ is a normed space with

$$\|x\| := \left( \sum_{j=1}^{+\infty} |x_j|^p \right)^{1/p}$$

Let $e_j := (0, \dots, 0, \underset{\substack{\uparrow \\ j\text{-th position}}}{1}, 0, \dots)$. Then $e_j \rightharpoonup 0$ but

$\|e_j\| = 1 \not\to 0$.

**DEFINITION** $(X, \|\cdot\|)$ normed space, $K \subseteq X$, $\tilde{K} \subseteq X^*$

1) $K$ is **COMPACT** if $\forall \{x_n\} \subseteq K$, $\exists x_0 \in K$ s.t. $x_{n_k} \to x_0$ along some subsequence
2) $K$ is **SEQUENTIALLY WEAKLY COMPACT** if $\forall \{x_n\} \subseteq K$, $\exists x_0 \in K$ s.t. $x_{n_k} \rightharpoonup x_0$ along some subsequence
3) $\tilde{K}$ is **SEQUENTIALLY WEAKLY* COMPACT** if $\forall \{\varphi_n\} \subseteq \tilde{K}$, $\exists \varphi_0 \in \tilde{K}$ s.t. $\varphi_{n_k} \overset{*}{\rightharpoonup} \varphi_0$ along some subsequence.

WARNING

If $(X, \tau)$ is a topological space then $K \subseteq X$ is compact if any OPEN COVER of $K$ admits a FINITE SUBCOVER.

If the topology $\tau$ is metrizable (e.g. metric or normed spaces) then SEQUENTIAL COMPACTNESS is equivalent to COMPACTNESS.

However, if $X$ is normed space, then the weak topology on $X$ and weak* topology on $X^*$ are NOT metrizable in general. Thus, in general WEAK (WEAK*) COMPACTNESS and WEAK (WEAK*) SEQUENTIAL COMPACTNESS are not equivalent. With additional assumptions, however, they are the same:

1) If $X$ is Banach then WEAK SEQUENTIAL COMPACTNESS and WEAK COMPACTNESS are equivalent (EBERLEIN - SMULIAN THEOREM)

2) If $X$ is a SEPARABLE BANACH space then WEAK* SEQUENTIAL COMPACTNESS and WEAK* COMPACTNESS are equivalent

**THEOREM (BANACH - ALAOGLU)** $(X, \|\cdot\|)$ normed space. Denote by $B := \{\varphi \in X^* \mid \|\varphi\|_{X^*} \leq 1\}$ the closed unit ball of $X^*$:

1) Then $B$ is WEAKLY* COMPACT

2) If in addition $X$ is BANACH and SEPARABLE then $B$ is also SEQUENTIALLY WEAKLY* COMPACT

There is a corollary of Banach-Alaoglu concering the weak compactness of the unit ball of $X$. For that we need the following definition

**REFLEXIVITY** $(X, \|\cdot\|)$ normed space. Consider $X^*$ and its dual WRT to the strong norm of $X^*$, i.e., $X^{**} := (X^*, \|\cdot\|_{X^*})^*$

Define the **CANONICAL EMBEDDING**

$$J : X \to X^{**} \quad \text{s.t.} \quad J(x)(\varphi) := \varphi(x), \quad x \in X, \varphi \in X^*$$

We have $\|J(x)\|_{X^{**}} = \|x\|_X$. We say that $X$ is **REFLEXIVE** if $J$ is surjective, i.e., if

$$X^{**} = \{ J(x), \; x \in X \}$$

**COROLLARY (of BANACH-ALAOGLU)**

$(X, \|\cdot\|)$ normed space. Define $B := \{ x \in X \mid \|x\| \leq 1 \}$.

1) If $X$ is Reflexive then $B$ is WEAKLY COMPACT
2) If $X$ is reflexive and Banach then $B$ is WEAKLY SEQUENTIALLY COMPACT

As a consequence of the PRINCIPLE OF UNIFORM BOUNDEDNESS (PUB) (see book of Conway), we have.

**PROPOSITION** $(X, \|\cdot\|)$ Banach space

1) If $\{x_n\} \subseteq X$ is s.t. $x_n \rightharpoonup x_0$ then $\sup_n \|x_n\| < +\infty$
2) If $\{\varphi_n\} \subseteq X^*$ is s.t. $\varphi_n \overset{*}{\rightharpoonup} \varphi_0$ then $\sup_n \|\varphi_n\|_{X^*} < +\infty$

Another important notion needed throughout the course is the one of lower semicontinuity.

**DEFINITION** $(X, d)$ metric space, $F: X \to \mathbb{R}$. We say that $F$ is LOWER SEMICONTINOUS at $x_0 \in X$ if

$$F(x_0) \leq \liminf_{n \to +\infty} F(x_n),$$

for all $\{x_n\} \subseteq X$ s.t. $x_n \to x_0$.

**DEFINITION** $(X, \|\cdot\|)$ normed space, $F: X \to \mathbb{R}$, $G: X^* \to \mathbb{R}$.

1) $F$ is (SEQUENTIALLY) WEAKLY LOWER SEMICONTINOUS at $x_0 \in X$ if

$$F(x_0) \leq \liminf_{n \to +\infty} F(x_n)$$

for all $\{x_n\} \subseteq X$ s.t. $x_n \rightharpoonup x_0$.

2) $G$ is (SEQUENTIALLY) WEAKLY$^*$ LOWER SEMICONTINOUS at $\varphi_0 \in X^*$ if

$$G(\varphi_0) \leq \liminf_{n \to +\infty} G(\varphi_n)$$

for all $\{\varphi_n\} \subseteq X^*$ s.t. $\varphi_n \overset{*}{\rightharpoonup} \varphi_0$.

**PROPOSITION** $(X, \|\cdot\|)$ normed space. Then

1) The norm $\|\cdot\|$ is WEAKLY SEQUENTIALLY LOWER SEMICONTINOUS, i.e.,

$$x_n \rightharpoonup x_0 \Rightarrow \|x_0\| \leq \liminf_{n \to +\infty} \|x_n\|$$

2) The norm $\|\cdot\|_{X^*}$ is WEAKLY$^*$ SEQUENTIALLY LOWER SEMICONTINOUS, i.e.,

$$\varphi_n \overset{*}{\rightharpoonup} \varphi_0 \Rightarrow \|\varphi_0\|_{X^*} \leq \liminf_{n \to +\infty} \|\varphi_n\|_{X^*}$$