MAT.494UB / SS 2021
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Calculus of Variations

Problem Sheet 7

Due date: 25.06.2021

Problem 7.1 (20 pts). Let (X,d) be a metric space, $f\colon X \to \overline{\mathbb{R}}$. Recall that $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and that the LSC Envelope of f is defined by

$$\bar{f}(x) := \inf \left\{ \liminf_{n \to +\infty} f(x_n) : \{x_n\} \subset X, \ x_n \to x \right\} .$$

(a) Prove that f is LSC at x_0 if and only if

$$f(x_0) = \bar{f}(x_0) .$$

(b) Let $g\colon X \to \overline{\mathbb{R}}$. Prove that

$$\overline{f+g} \geq \bar{f} + \bar{g} .$$

(c) Let $g\colon X \to \mathbb{R}$ be continuous. Prove that

$$\overline{f+g} = \bar{f} + g .$$

(d) Find one example of metric space (X,d) and functions $f, g\colon X \to \overline{\mathbb{R}}$ such that

$$\overline{f+g} > \bar{f} + \bar{g}$$

for at least one $x \in X$.

Hint: (c) Use Propositions 10.15 and 10.8 in the Lecture Notes.

Problem 7.2 (20 pts). Let $X := L^2(a,b)$ equipped with the metric d induced by the L^2 norm. Define $F, G\colon X \to \overline{\mathbb{R}}$ by

$$F(u) := \begin{cases} \int_a^b \dot{u}^2 \, dx & \text{if } u \in C^1[a,b] \\ +\infty & \text{if } u \in X \smallsetminus C^1[a,b] \end{cases} \qquad G(u) := \begin{cases} \int_a^b \dot{u}^2 \, dx & \text{if } u \in H^1(a,b) \\ +\infty & \text{if } u \in X \smallsetminus H^1(a,b) \end{cases}$$

Prove that $\bar{F} = G$ by using Proposition 10.18 in the Lecture Notes applied with X, d, $f = F$, $g = G$ and $D = C^1[a,b]$.

Hint: You may find the following results in the Lecture Notes useful: Theorem 7.27, Theorem 7.24, Corollary 7.10.

Problem 7.3 (20 pts). Let (X, d) be a metric space, $f_n, f\colon X \to \overline{\mathbb{R}}$. Suppose that

(i) $f_n \to f$ uniformly on compact sets of X,

(ii) f is LSC.

Prove that $f_n \to f$ in the sense of Γ-convergence.

Hint: If $x_n \to x_0$ then the set $K := \{x_n,\, n \in \mathbb{N}\} \cup \{x_0\}$ is compact in (X, d).

Problem 7.4 (20 pts).

(a) (X, d) metric space. Suppose that $f_n, f\colon X \to \overline{\mathbb{R}}$ are such that $f_n \to f$ in the sense of Γ-convergence. Moreover assume that $g_n, g\colon X \to \mathbb{R}$ are such that

 (i) $g_n \to g$ uniformly on compact sets of X,

 (ii) g is continuous.

 Prove that

$$f_n + g_n \to f + g$$

 in the sense of Γ-convergence.

(b) Let $X = \mathbb{R}$. Define

$$f_n(x) := \arctan(nx) + \frac{x^2}{n} \qquad \text{for} \quad x \in \mathbb{R}\,.$$

 Compute the Γ-limit of f_n in $\mathbb{R}$ as $n \to +\infty$.

Problem 7.5 (20 pts). Let $a < b$ and $A_n\colon [a, b] \to \mathbb{R}$ be a sequence of functions such that

(i) $A_n \geq 0$ a.e. in (a, b), for all $n \in \mathbb{N}$,

(ii) $A_n \leq M$ a.e. in (a, b), for all $n \in \mathbb{N}$, for some $M > 0$,

(iii) $A_n \rightharpoonup A$ weakly in $L^2(a, b)$.

Define the functionals $F_n, F\colon L^2(a, b) \to \overline{\mathbb{R}}$ by

$$F_n(u) := \begin{cases} \int_a^b \dot{u}^2 + A_n u^2 \, dx & \text{if } u \in C^1[a, b] \\ +\infty & \text{otherwise} \end{cases} \qquad F(u) := \begin{cases} \int_a^b \dot{u}^2 + A u^2 \, dx & \text{if } u \in H^1(a, b) \\ +\infty & \text{otherwise} \end{cases}$$

Prove that $F_n \to F$ in the sense of Γ-convergence with respect to the $L^2(a, b)$ metric.

Hint: You may find the following results in the Lecture Notes useful: Theorem 7.27, Theorem 7.24.