MAT.494UB / SS 2021
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Calculus of Variations

Problem Sheet 6

Due date: 11.06.2021

Problem 6.1 (20 pts) - Assumptions of Theorem 9.9 are optimal.

Define the functional $F\colon W_0^{1,4}(0,1) \to \mathbb{R}$ by

$$F(u) := \int_0^1 (\dot{u}^2 - 1)^2 + u^4 \, dx \,,$$

and set

$$m := \inf\{F(u) : \; u \in W_0^{1,4}(0,1)\} \,.$$

(a) Prove that $m = 0$.

(b) Prove that F admits no minimizers in $W_0^{1,4}(0,1)$.

(c) Does the non existence of a minimizer contradict Theorem 9.9 in the Lecture Notes?

Hint: (a) Define a suitable bounded function $s\colon [0,1] \to \mathbb{R}$ such that $s(0) = s(1) = 0$ and $|s'| = 1$ a.e. in $(0,1)$. Then define a sequence $\{u_n\} \subset W_0^{1,4}(0,1)$ by suitably rescaling s, and show that $F(u_n) \to 0$.

Problem 6.2 (20 pts) - Assumptions of Theorem 9.9 are optimal.

Define the space

$$X := \{u \in W^{1,1}(0,1) : \; u(0) = 0, \; u(1) = 1\}$$

and the functional $F\colon W^{1,1}(0,1) \to \mathbb{R}$ by

$$F(u) := \int_0^1 \sqrt{u^2 + \dot{u}^2} \, dx \,.$$

Set

$$m := \inf\{F(u) : \; u \in X\} \,.$$

(a) Prove that $m = 1$.

(b) Prove that F admits no minimizers in X.

(c) Does the non existence of a minimizer contradict Theorem 9.9 in the Lecture Notes?

Hint: (a), (b) Can be done with similar ideas to the ones used for the minimization of the functionals G and H in the "Three Examples" of Lesson 3. Also recall that Sobolev functions are continuous.

Problem 6.3 (30 pts). Define the space

$$X := \{u \in W^{1,2}(0,2) : \ u(0) = 0, \ u(2) = 3\}$$

and the functional $F \colon W^{1,2}(0,2) \to \mathbb{R}$ by

$$F(u) := \int_0^2 \frac{1}{2}\,\dot{u}^2 + g(u)\,dx\,, \qquad g(s) := \int_0^s \arctan(t)\,dt\,.$$

(a) Verify the assumptions of Theorem 9.9 in the Lecture Notes to prove that F admits at least one minimizer over X.
(b) Is the minimizer unique?
(c) Prove that if $u_0 \in X$ minimizes F over X, then $u_0 \in C^\infty([0,2])$.

Hint: (b) Use Theorem 9.9 and Theorem 5.2 in the Lecture Notes. (c) Write down the weak ELE (using Theorem 8.4 in the Lecture Notes) and then use the bootstrap argument employed in Example 9.8 in the Lecture Notes.

Problem 6.4 (30 pts) - The Brachistochrone problem. The problem of the brachistochrone, formulated by Galileo in 1638, had a very strong influence on the development of the calculus of variations. It was resolved by John Bernoulli in 1696 and almost immediately after also by James, his brother, Leibniz and Newton. A decisive step was achieved with the work of Euler and Lagrange who found a systematic way of dealing with problems in this field by introducing what is now known as the Euler–Lagrange equation.
The aims to find the shortest path between two points that follows a point mass moving under the influence of gravity.

a) First, solve the physical problem by expressing the running time of the point mass in terms of its trajectory. Let the starting and ending points of the trajectory be the origin $(0,0)$ and (b,B) of a coordinate system whose y-axis points downwards (see Figure 1). This gives the functional to be minimized.

b) Compute the first variation on an interval $[\delta, b]$ where δ is small. Then the trajectory $\tilde{y}$ (parametrized by $x \in [0,b]$) solves the Euler–Lagrange equation piecewise on $[\delta, b]$. Show also that $\tilde{y}$ solves the Euler–Lagrange equation in (0,b] and $\tilde{y} \in C^2(0,b)$. Then you can find a differential equation, whose solution is the cycloid.

b) We skip the proof that the cycloid gives indeed the minimal value. However, for $B = 2\pi/b$, compare the running time of a point mass acted on by gravity on the line segment and on the cycloid from $(0,0)$ to (b,B). Compute the ratio of the running times.

Hint: See H. Kielhoefer, Calculus of Variations: An Introduction to the One-dimensional Theory with Examples and Exercises, 2018. Texts in Applied Mathematics, vol. 67, Springer.

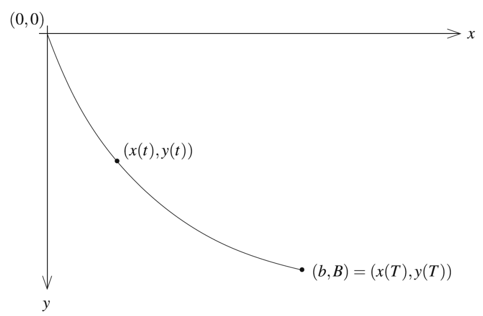

Figure 1: Coordinate system whose y-axis points downwards. The starting and ending points of the trajectory are the origin $(0,0)$ and (b,B), respectively.