MAT.494UB / SS 2021
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Calculus of Variations

Problem Sheet 5

Due date: 28.05.2021

Problem 5.1 (20 pts).

(a) Let $-\infty < a < b < +\infty$ and $u \in C^1([a,b])$. Prove that $u \in W^{1,p}(a,b)$ for all $1 \le p \le +\infty$.

(b) Define $u(x) := |x| + 1$. Prove that $u \in W^{1,p}(-1,1)$ for all $1 \le p \le +\infty$.

(c) Define

$$u(x) := \begin{cases} 1 & \text{if } x > 0\,, \\ -1 & \text{if } x < 0\,. \end{cases}$$

Prove that $u \notin W^{1,p}(-1,1)$ for any $1 \le p \le +\infty$.

Hint: (c) Argue by contradiction and consider a sequence of Cut-off functions around the origin with supports converging to $\{0\}$.

Problem 5.2 (30 pts) - Characterization of Sobolev functions.

(a) Let $I \subset \mathbb{R}$ be open and $1 < p \le +\infty$. Assume that $u \in L^p(I)$. Prove that they are equivalent:

  (i) $u \in W^{1,p}(I)$,

  (ii) There exists a constant $C > 0$ such that

$$\left| \int_I u\varphi' \, dx \right| \le C \, \|\varphi\|_{L^{p'}} \qquad \text{for all} \quad \varphi \in C_c^1(I)\,,$$

  where $p' = p/(p-1)$.

(b) Let $I \subset \mathbb{R}$ be open and $p = 1$. Does the equivalence between (i) and (ii) in point (a) hold?

Hint: (a) For the implication (ii) $\implies$ (i), consider the functional $T \colon C_c^1(I) \to \mathbb{R}$ defined by $T(\varphi) := \int_I u\varphi' \, dx$. Can it be extended to $L^{p'}(I)$?

Problem 5.3 (30 pts) - Another characterization of Sobolev functions.

Let $1 < p < +\infty$ and $u \in L^p(\mathbb{R})$. Prove that they are equivalent:

(I) $u \in W^{1,p}(\mathbb{R})$,

(II) There exists a constant $C > 0$ such that

$$\|\tau_h u - u\|_{L^p} \le C|h| \qquad \text{for all} \quad h \in \mathbb{R}\,,$$

where $(\tau_h u)(x) := u(x+h)$.

Hint: For (I) $\implies$ (II) use that if $u \in W^{1,p}(\mathbb{R})$ then $u(y) - u(x) = \int_x^y u'(t)\,dt$ for a.e. $x, y \in \mathbb{R}$. For (II) $\implies$ (I), show that (II) implies (ii) in Exercise 5.2. The following identity may be useful: for all $\varphi \in C_c^1(\mathbb{R}), u \in L^p(\mathbb{R})$ one has $\int_{\mathbb{R}} u(x)[\varphi(x+h) - \varphi(x)]\,dx = \int_{\mathbb{R}} [u(x-h) - u(x)]\varphi(x)\,dx$.

Problem 5.4 (20 pts) - Generalized Poincaré Inequality. Let $I = (a, b)$ be bounded and $1 \leq p < +\infty$. Let $V \subset W^{1,p}(I)$ be a subspace such that

(i) V is closed in $W^{1,p}(I)$,

(ii) If $u \in V$ is constant, then $u \equiv 0$.

Prove that there exists a constant $C > 0$ such that

$$\|u\|_{W^{1,p}} \leq C \|u'\|_{L^p} \qquad \text{for all} \quad u \in V\,.$$

Hint: Follow the lines of the Proof by Contradiction of Theorem 7.35 in the Lecture Notes.