MAT.494UB / SS 2021
Dr. Cinzia Soresina
Dr. Silvio Fanzon

Institute for Mathematics
and Scientific Computing

Calculus of Variations

Problem Sheet 4

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We start with a few definitions which are useful for the following exercises.

$\mathbf{L^p}$ spaces: Let $I \subset \mathbb{R}$ be open, and $1 \leq p < +\infty$. We define

$$L^p(I) := \left\{ u \colon \Omega \to \mathbb{R} : \; u \text{ Lebesgue measurable}, \int_I |u|^p \, dx < +\infty \right\} ,$$

where dx denotes the one-dimensional Lebesgue measure on $\mathbb{R}$. Recall that $L^p(I)$ is a Banach space with the norm

$$\|u\|_p := \left(\int_I |u|^p \, dx \right)^{1/p} .$$

Similarly, we define the space of locally integrable functions

$$L^p_{\mathrm{loc}}(I) := \{ u \colon \Omega \to \mathbb{R} : \; u \chi_K \in L^p(I) \;\text{ for all compact sets }\; K \subset I \} ,$$

where χ_K is the characteristic function of K, i.e., $\chi_K(x) = 1$ if $x \in K$ and $\chi_K(x) = 0$ if $x \notin K$.

Convolutions: Given two measurable functions $u, v \colon \mathbb{R} \to \mathbb{R}$, their convolutions is defined as

$$(u \star v)(x) := \int_{\mathbb{R}} u(x-y) v(y) \, dy ,$$

for all $x \in \mathbb{R}$ such that the right-hand side is well-defined. Note that, as soon as the right-hand side is finite, we have $u \star v = v \star u$. The following result gives sufficient conditions for the convolution to be well-defined.

Young's Theorem. Let $1 \leq p \leq +\infty$ and $u \in L^1(\mathbb{R})$, $v \in L^p(\mathbb{R})$. The for a.e. $x \in \mathbb{R}$ the map $y \mapsto u(x-y)v(y)$ belongs to $L^1(\mathbb{R})$, so that $(u \star v)(x)$ is finite. Moreover $u \star v \in L^p(\mathbb{R})$ and

$$\|u \star v\|_p \leq \|u\|_1 \, \|v\|_p .$$

Mollifiers: A family of mollifiers $\{\rho_\varepsilon\}_{\varepsilon > 0}$ is any family of functions $u_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ such that

$$\rho_\varepsilon \in C^\infty_c(\mathbb{R}) , \quad \operatorname{supp} \rho_\varepsilon \subset [-\varepsilon, \varepsilon] , \quad \rho_\varepsilon \geq 0 \;\text{ on }\; \mathbb{R} , \quad \int_{\mathbb{R}} \rho_\varepsilon(x) \, dx = 1 , \tag{1}$$

for all $\varepsilon > 0$. Sometimes it is more convenient to deal with a sequence, rather than a family indicized by ε: In this case the mollifiers are denoted by $\{\rho_n\}_{n \in \mathbb{N}}$, with ρ_n satisfying (1) with $\varepsilon = 1/n$.

Standard Mollifiers: Let $\rho \colon \mathbb{R} \to \mathbb{R}$ be given by

$$\rho(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1} \right) & \text{if } |x| < 1 , \\ 0 & \text{if } |x| \geq 1 , \end{cases}$$

where $C > 0 := 1/\int_{\mathbb{R}} \rho$. Notice that ρ satisfies

$$\rho \in C_c^\infty(\mathbb{R}), \quad \rho \geq 0\,, \quad \operatorname{supp} \rho \subset [-1,1]\,, \quad \int_{\mathbb{R}} \rho(x)\,dx = 1\,.$$

With ρ we can define the standard family of mollifiers

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon}\,\rho\left(\frac{x}{\varepsilon}\right)\,.$$

Notice that $\{\rho_\varepsilon\}_{\varepsilon>0}$ satisfies (1). If we want instead a sequence of standard mollifiers, just set

$$\rho_n(x) := n\,\rho(nx)\,.$$

Notation: Let $a < b$. Recall that the space $C^1([a,b])$ is normed by $\|u\| := \|u\|_\infty + \|u'\|_\infty$. Moreover we introduce the space of piecewise C^1 functions

$$C^1_{\text{pw}}([a,b]) := \big\{u \in C([a,b]) :\ \exists\ N \in \mathbb{N},\ a = p_1 < p_2 < \ldots < p_N = b \ \text{ s.t. } u \in C^1[p_i, p_{i+1}],\ \ \forall\ i = 1, \ldots, N-1\big\}$$

Notice that the above partition can change depending on u. The map

$$\|u\|_{\text{pw}} := \|u\|_\infty + \max_{i=1,\ldots,N-1}\ \max_{x\in[p_1,p_{i+1}]} |u'(x)|$$

defines a norm over $C^1_{\text{pw}}([a,b])$.

Problem 4.1 (20 pts) - Smoothing by convolution.

a) Let $u \in L^1_{\text{loc}}(\mathbb{R})$ and $v \in C_c(\mathbb{R})$. Prove that $u \star v$ is well defined and $u \star v \in C(\mathbb{R})$.

b) Let $k \geq 1$, $u \in L^1_{\text{loc}}(\mathbb{R})$ and $v \in C_c^k(\mathbb{R})$. Prove that $u \star v \in C^k(\mathbb{R})$ with

$$\frac{d^k}{dx^k}(u \star v) = u \star v^{(k)}\,.$$

Hint: (a) Let $x_n \to x_0$. Since v is compactly supported, there exists some compact set $K \subset \mathbb{R}$ such that $(x_n - \operatorname{supp} v) \subset K$ for all $n \in \mathbb{N}$. (b) You can just show this for $k = 1$. The case $k > 1$ follows trivially by induction. Also notice that v' is uniformly continuous in $\mathbb{R}$, since $\operatorname{supp} v' \subset \operatorname{supp} v$, and $\operatorname{supp} v$ is compact. Moreover it may be useful to recall the Fundamental Theorem of Calculus, namely, $v(x+t) - v(x) = \int_0^1 v'(x+ts)\,t\,ds$.

Problem 4.2 (30 pts) - Cut-off. The goal of this exercise is to construct a cut-off function like the one introduced in Remark 3.3 in the lecture Notes. This is requested in point (b).

(a) Let $u \in L^1(\mathbb{R})$, $g \in L^p(\mathbb{R})$ for some $1 \leq p \leq +\infty$. Show that

$$\operatorname{supp}(u \star v) \subset \overline{\operatorname{supp} u + \operatorname{supp} v}\,.$$

(b) Let $K \subset \mathbb{R}$ be compact and $\varepsilon > 0$ fixed. Construct a function $\eta_\varepsilon \in C_c^\infty(\mathbb{R})$ such that:

i) $0 \leq \eta_\varepsilon \leq 1$ on $\mathbb{R}$,

ii) $\eta_\varepsilon(x) = 1$ for all $x \in K$,

iii) $\eta_\varepsilon(x) = 0$ for all $x \in \mathbb{R} \setminus K_\varepsilon$, with $K_\varepsilon := K + (-\varepsilon, \varepsilon)$,

iv) $|\eta_\varepsilon^{(k)}(x)| \leq C_k\,\varepsilon^{-k}$, for all $x \in \mathbb{R}$, $k \in \mathbb{N}$, with $C_k > 0$ constant depending only on k.

Hint: (a) $u \star v$ is well-defined by Young's Theorem stated above. (b) Take $\eta_\varepsilon := \chi_{K_{\varepsilon/2}} \star \rho_{\varepsilon/2}$, with ρ_ε the standard mollifier. Of course you can invoke Exercise 4.1 to prove smoothness and compute derivatives.

Problem 4.3 (25 pts) - Rounding corners. The goal of this exercise is to make rigorous the procedure of "rounding the corner" which has been used several times during the course.
Consider the functional $F\colon X \to \mathbb{R}$

$$F(u) := \int_0^1 \sqrt{|\dot{u}(x)|}\, dx\,, \qquad X := \{u \in C^1([0,1]) :\ u(0) = 2,\ u(1) = 0\}\,.$$

a) Let $a < x_0 < b$, $u \in C([a,b])$ such that $u \in C^1([a,x_0])$, $u \in C^1([x_0,b])$. Thus $u \in C^1_{\text{pw}}([a,b])$. Let $\delta > 0$ be such that $I_\delta := (x_0 - \delta, x_0 + \delta) \subset (a,b)$. Using Exercise 4.2 point (b), construct $\tilde{u} \in C^1([a,b])$ such that

$$\tilde{u} = u \ \text{ in } \ [a,b] \smallsetminus I_\delta\,, \qquad \|\tilde{u}\| \le \left(c_1 + c_2\,\frac{1}{\delta}\right) \|u\|_{\text{pw}}\,,$$

where $c_1, c_2 > 0$ are constants not depending on δ and u.

b) Let $0 < \varepsilon < \sqrt{2} - 1$. Define $u_\varepsilon\colon [0,1] \to \mathbb{R}$ by

$$u_\varepsilon(x) := \begin{cases} 2 - \dfrac{x}{\varepsilon} & \text{if} \quad 0 \le x \le 2\varepsilon\,, \\ 0 & \text{if} \quad 2\varepsilon \le x \le 1\,. \end{cases}$$

Note that $u_\varepsilon \in C([0,1])$, $u_\varepsilon \in C^1([0,2\varepsilon])$ and $u_\varepsilon \in C^1([2\varepsilon,1])$. Let $\tilde{u}_\varepsilon$ be constructed by applying point (a) to u_ε with $a = 0$, $b = 1$, $x_0 := 2\varepsilon$ and $\delta := \varepsilon^2$. Prove that

i) $\tilde{u}_\varepsilon \in X$,

ii) $m = 0$, where $m := \inf\{F(u) :\ u \in X\}$,

iii) F admits no minimizer over X.

Problem 4.4 (15 pts) - Approximation Result. The goal of this exercise is to prove Remark 3.5 from the Lecture Notes (in the case $(a,b) = \mathbb{R}$).

a) Let ρ_n be a sequence of standard mollifiers and $u \in C(\mathbb{R})$. Prove that $\rho_n \star u \to u$ uniformly on compact sets, i.e., for all $K \subset \mathbb{R}$ compact it holds

$$\lim_{n\to+\infty} \max_{x \in K} |(\rho_n \star u)(x) - u(x)| = 0\,.$$

b) Let $u \in C_c(\mathbb{R})$. Use point (a) to construct a sequence $u_n \in C_c^\infty(\mathbb{R})$ such that

i) $u_n \to u$ uniformly on compact sets of $\mathbb{R}$,

ii) u_n is uniformly bounded, i.e.,

$$\sup_{n \in \mathbb{N}} \|u_n\|_\infty < +\infty\,.$$

Hint: You can use Exercise 4.1 and Exercise 4.2 point (b).

Problem 4.5 (10 pts). Consider the functional $F\colon X \to \mathbb{R}$

$$F(u) := \int_0^1 \exp(-\dot{u}^2)\, dx\,, \qquad X := \{u \in C^1([0,1]) : u(0) = u(1) = 0\}\,.$$

a) Prove that $u_0 \equiv 0$ is stationary for F.

b) Prove that $u_0 \equiv 0$ is a maximum point of F in X.

c) Let $m := \inf\{F(u) :\ u \in X\}$. Prove that $m = 0$ and that F admits no minimizer over X.

Hints: (a) Use Theorem 4.5 from the Lecture Notes to compute the first variation. (c) Consider $u_n(x) := n\,(x - 1/2)^2 - n/4$.