MAT.494UB / SS 2021
Dr. Cinzia Soresina
Dr. Silvio Fanzon

Institute for Mathematics
and Scientific Computing

Calculus of Variations

Problem Sheet 3

Due date: 23.04.2021

Throughout the exercise paper whenever we say differentiable, we mean Fréchet differentiable.

Problem 3.1 (15 pts). In this exercise we show that the Fréchet derivative is linear and that the chain rule and product rule hold.

a) (Linearity) Let X, Y be normed spaces, $U \subset X$ open, $\alpha \in \mathbb{R}$, and $F, G \colon U \to Y$ be differentiable at $u_0 \in U$. Prove that $\alpha F + G$ is differentiable at u_0 with $(\alpha F + G)'(u_0) \in \mathcal{L}(X, Y)$ given by

$$(\alpha F + G)'(u_0) = \alpha F'(u_0) + G'(u_0)\,.$$

b) (Chain Rule) Let X, Y, Z be normed spaces, $U \subset X$, $V \subset Y$ open sets. Let $F \colon U \to V$, $G \colon V \to Z$. Assume that F and G are differentiable at $u_0 \in U$ and at $F(u_0)$, respectively. Then $G \circ F \colon U \to Z$ is differentiable at u_0, with $(G \circ F)'(u_0) \in \mathcal{L}(X, Z)$ given by the composition of linear continuous operators

$$(G \circ F)'(u_0) = G'(F(u_0)) \circ F'(u_0)\,.$$

c) (Product Rule) Let X be a normed space, $U \subset X$ open and $F, G \colon U \to \mathbb{R}$ be differentiable at $u_0 \in U$. Show that the product function FG is differentiable at u_0 with $(FG)'(u_0) \in X^*$ given by

$$(FG)'(u_0) = G(u_0)F'(u_0) + F(u_0)G'(u_0)\,.$$

Problem 3.2 (15 pts). Let H be a Hilbert space with induced norm $\|\cdot\|$. Define $F, G \colon H \to \mathbb{R}$ by $F(x) := \|x\|^2$, $G(x) := \|x\|$.

a) Show that F is differentiable in H and that $F \in C^1(H)$.

b) Show that G is differentiable for all $x \in H \setminus \{0\}$ but not differentiable at $x = 0$.

   Hint: Chain rule for the first part of the statement. By contradiction for the second.

c) Find an example of a normed space X such that $G(x) := \|x\|$ is not differentiable in $X \setminus \{0\}$.

Problem 3.3 (15 pts). Let H be a Hilbert space and $a \colon H \times H \to \mathbb{R}$ be bilinear, symmetric and continuous, that is, there exists $M > 0$ such that $|a(x, y)| \le M \|x\| \|y\|$ for all $x, y \in H$. Let $b \in H$ and define the map $F \colon H \to \mathbb{R}$ by

$$F(u) := Q(u) + L(u), \quad Q(u) := \frac{1}{2} a(u, u), \quad L(u) := \langle b, u \rangle\,.$$

a) Prove that $F \in C^1(H)$, with derivative

$$F'(u)(v) = a(u, v) + L(v)\,.$$

b) Prove that $F \in C^2(H)$, with
$$F''(u)(v_1, v_2) = a(v_1, v_2) \, .$$

c) Suppose that a is only bilinear and continuous (not symmetric). Compute F', F'' in this case.

Problem 3.4 (15 pts). Let X be a normed space, $U \subset X$ be open. Let $F \colon U \to \mathbb{R}$. We say that $\hat{u} \in U$ is a local minumum for F if there exists a neighbourhood V of $\hat{u}$ such that

$$F(\hat{u}) \leq F(u) \quad \text{for all} \quad u \in V\,.$$

a) Suppose that F is differentiable at $\hat{u}$. Show that

$$F'(\hat{u}) = 0\,.$$

b) Suppose that F is differentiable in U and twice differentiable at $\hat{u}$. Prove that

$$F''(\hat{u})(v,v) \geq 0 \quad \text{for all} \quad v \in X\,.$$

Problem 3.5 (40 pts). Consider the functionals in $C^1([0,3])$

$$F(u) = \int_0^3 \dot{u}^2\,dx, \qquad G(u) = \int_0^3 (\dot{u}^2 + u^2)\,dx, \qquad H(u) = \int_0^3 (\dot{u}^2 - 6u)\,dx\,.$$

(A) For the above functionals:

(A1) Compute the first variation of F, G, H at $u \in C^1([0,3])$ in direction $v \in C^1([0,1])$.

(A2) Define

$$X = \{u \in C^1([0,3]) : u(0) = 2,\ u(3) = 6\}\,.$$

For $u \in C^2([0,3]) \cap X$, integrate by parts the first variation of F, G, H. After that, characterize all the stationary points of F, G, H in $C^2([0,3]) \cap X$.

(A3) Verify that the found stationary points are the unique minimizers for F, G, H.

(B) Study the minimum problem for F, G and H in the following sets

- $X_1 = \{u \in C^1([0,3]) : u(0) = 2\}$,
- $X_2 = \{u \in C^1([0,3])\}$,
- $X_3 = \{u \in C^1([0,3]) : u(0) = 2,\ u(3) = 6,\ \int_0^3 u\,dx = 1\}$.

Determine if the problem has a solution or not. If the minimum exists, characterize all minimizers (for G in the case of X_3, it is ok not to compute the exact coefficients).