MAT.494UB / SS 2021
Dr. Cinzia Soresina
Dr. Silvio Fanzon

Institute for Mathematics
and Scientific Computing

Calculus of Variations

Problem Sheet 2

Due date: 26.03.2021

Problem 2.1 (15 pts). Let (X, d) be a metric space.

a) Let $C \subset X$. Show that C is closed if and only if

$$\{x_n\} \subset C,\, x_n \to x_0 \text{ implies } x_0 \in C\,.$$

b) Let $A \subset X$. Recall that the closure of A is defined by $\overline{A} := \cap\{C \subset X :\ A \subset C,\ C \text{ closed }\}$. Define the set $L := \{x \in X : \exists\, \{x_n\} \subset A \text{ s.t. } x_n \to x\}$. Prove that $\overline{A} = L$.

Hint: Show that L is closed by means of a diagonal argument.

Problem 2.2 (15 pts). Let (X, d_X), (Y, d_Y) be metric spaces and assume that X is compact.

a) Let $\{x_n\} \subset X$ be a sequence having the property that every convergent subsequence converges to the same point x_0. Prove that $x_n \to x_0$ as $n \to \infty$.

b) Suppose that $F \colon X \to Y$ is continuous. Show that $F(X)$ is compact in Y.

c) Suppose that $F \colon X \to Y$ is continuous. Show that F is also uniformly continuous, that is, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(F(x_1), F(x_2)) < \varepsilon$, for all $x_1, x_2 \in X$ satisfying $d_X(x_1, x_2) < \delta$.

Problem 2.3 (15 pts). Let (X, d) be a compact metric space and $F \colon X \to \mathbb{R}$ be lower semicontinuous. Prove that:

a) F is bounded from below, that is, there exists $m \in \mathbb{R}$ such that $F(x) \geq m$ for all $x \in X$,

b) F admits minimum, that is, there exists $\hat{x} \in X$ such that $F(\hat{x}) = \inf_{x \in X} F(x)$.

Hint: Consider a minimizing sequence, i.e., $\{x_n\} \subset X$ such that $F(x_n) \to \inf_{X \in X} F(x)$ (by definition of infimum it always exists). What happens to x_n to the limit?

Problem 2.4 (25 pts). Let X be a real normed space, and $\Lambda_n, \Lambda \in X^*$, $x_n, x \in X$ for $n \in \mathbb{N}$.

a) Show the following implications between convergences in X^*:

   i) if $\Lambda_n \to \Lambda$ strongly then $\Lambda_n \rightharpoonup \Lambda$ weakly,

   ii) if $\Lambda_n \rightharpoonup \Lambda$ weakly then $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ weakly*.

   Hint: Use the canonical embedding $J \colon X \to X^{**}$.

b) Assume in addition that X is reflexive. Prove that in X^* we have that $\Lambda_n \rightharpoonup \Lambda$ weakly if and only if $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ weakly*.

   Hint: In this case, by definition, the canonical embedding is surjective.

c) Prove that weak* limits in X^* and weak limits in X are unique.

Hint: Use one of the corollaries of the Hahn-Banach Theorem (Conway III.6.8, Pag 79).

d) Assume that $x_n \rightharpoonup x$ weakly in X. Show that x_n is norm bounded, that is, $\sup_n \|x_n\|_X < \infty$.

Hint: Use the canonical embedding and the Principle of Uniform Boundedness (Conway III.14.1, Pag 95).

e) Assume in addition that X is a Banach space. Show that if $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ then $\sup_n \|\Lambda_n\|_{X^*} < \infty$.

Hint: Use the Principle of Uniform Boundedness.

Problem 2.5 (30 pts). Let H be a real Hilbert space, $x_n, x, y_n, y \in H$ for $n \in \mathbb{N}$.

a) Show that $x_n \to x$ strongly in H if and only if

$$x_n \rightharpoonup x \quad \text{weakly in} \quad H \quad \text{and} \quad \|x_n\| \to \|x\| \,.$$

b) Prove that the norm is weakly lower semicontinuous, that is, $x_n \rightharpoonup x$ implies

$$\|x\| \leq \liminf_{n \to +\infty} \|x_n\| \,.$$

c) Assume that $x_n \to x$ and $y_n \rightharpoonup y$. Prove that

$$\lim_{n \to +\infty} \langle x_n, y_n \rangle = \langle x, y \rangle \,.$$

d) Let $W \subset H$ be such that $\operatorname{span}(W)$ is dense in H with respect to the induced norm. Suppose that $\sup_n \|x_n\| \leq M$ for all $n \in \mathbb{N}$ and that there exists $x \in H$ such that

$$\lim_{n \to +\infty} \langle x_n, w \rangle = \langle x, w \rangle \qquad \text{for all} \quad w \in W \,. \tag{1}$$

Prove that $x_n \rightharpoonup x$.

e) Find a counterexample to prove that the boundedness assumption for the sequence $\{x_n\}$ in point (d) is necessary to have weak convergence, i.e., construct a sequence $\{x_n\}$ such that $\sup_n \|x_n\| = +\infty$, and that (1) holds for some $x \in H$, but $x_n \not\rightharpoonup x$.

Hint: Assume that H is separable and take $W = \{e_n\}$ basis. With it, construct a sequence satisfying the required properties.