MAT.495UB / SS 2021
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Calculus of Variations

Problem Sheet 1

Due date: 12.03.2021,8am

Problem 1.1 (30 pts). Let (X, d) be a metric space.

a) Let $\{x_n\} \subset X$ be a sequence such that $x_n \to x$ as $n \to \infty$, for some $x \in X$. Show that $\{x_n\}$ is a Cauchy sequence.

b) Suppose in addition that X is a real vector space and that the distance d satisfies:

   i) $d(x + a, y + a) = d(x, y)$ for all $a, x, y \in X$ (translation invariance),

   ii) $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $x, y \in X$, $\lambda \in \mathbb{R}$ (one-homogeneity).

   Prove that $\|x\| := d(x, 0)$ defines a norm over X.

c) Define a metric on a real vector space X which does not satisfy either (i) and/or (ii).

Problem 1.2 (15 pts). Let X be a real vector space, such that $X \neq \{0\}$. Show that there exists at least one norm on X.

Hint: every real vector space X has an algebraic basis, that is, there exists $B = \{e_i,\ i \in I\} \subset X$ such that every $x \in X$ with $x \neq 0$ can be uniquely written as $x = \sum_{j=1}^{n} \lambda_{i_j} e_{i_j}$ for some $n \in \mathbb{N}$, $\lambda_{i_j} \in \mathbb{R} \setminus \{0\}$ and $i_j \in I$ pairwise distinct for $j = 1, \ldots, n$. Use this fact to define a norm over X.

Remember: For a metric space (X, d) the collection of sets

$$\tau := \{A \subset X : \forall x \in X, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subset A\}$$

is called the topology induced by d over X, where $B_\varepsilon(x) := \{y \in X : d(x, y) < \varepsilon\}$. The sets $A \in \tau$ are called open. A set $C \subset X$ is closed if $C^c := X \setminus C$ is open.

Problem 1.3 (30 pts). The aim of this exercise is to show the difference between metrics and norms.

a) Let X be a real vector space. Suppose that $\|\cdot\|_1, \|\cdot\|_2 : X \to \mathbb{R}$ are norms on X which induce the same topology τ. Prove that $(X, \|\cdot\|_1)$ is complete if and only if $(X, \|\cdot\|_2)$ is complete.

   Hint: Consider the identity map $I \colon (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$. Is it a linear and bounded operator?

b) Let $d_1, d_2 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that $d_1(x, y) := |x - y|$, $d_2(x, y) := |\varphi(x) - \varphi(y)|$, with $\varphi(x) := x/(1 + |x|)$. We know that $(\mathbb{R}, d_1)$ is a complete metric space. Prove that:

   i) d_2 is a metric over $\mathbb{R}$;

   ii) d_1 and d_2 induce the same topology τ over $\mathbb{R}$;

   iii) $(\mathbb{R}, d_2)$ is not complete.

   iv) Does there exist a norm $\|\cdot\|_2$ on $\mathbb{R}$ such that $\|x - y\|_2 = d_2(x, y)$ for all $x, y \in \mathbb{R}$?

Problem 1.4 (25 pts). Let (X, d_X), (Y, d_Y) be metric spaces and denote by τ_X, τ_Y the respective induced topologies. Recall that a map $F\colon X \to Y$ is continuous if $F^{-1}(A) \in \tau_X$ for all $A \in \tau_Y$, where $F^{-1}(A) := \{x \in X : \exists\, y \in A \text{ s.t. } F(x) = y\}$. Show that they are equivalent:

i) F is continuous,

ii) For all $x_0 \in X$ it holds: for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(F(x), F(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$ (here δ depends on x_0),

iii) F is sequentially continuous, that is, for all $x_0 \in X$ and $\{x_n\} \subset X$ such that $d_X(x_n, x_0) \to 0$, it holds $d_Y(F(x_n), F(x_0)) \to 0$.

Hint: It may be easier to show that (i) is equivalent to (ii), and (ii) is equivalent to (iii).