Institute for Mathematics
and Scientific Computing

Advanced Functional Analysis

Problem Sheet 6

Due date: January 20, 2020

You are required to present Problems 6.2, 6.4, 6.5, 6.6, 6.7, 6.9. The rest of the problems will not be marked, but I recommend doing them as a preparation for the final exam.

I will refer to the following books:

- Haim Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, 2011, Springer-Verlag New York.
- John B. Conway, *A Course in Functional Analysis* (Second Edition), 1990, Springer.
- Walter Rudin, *Functional Analysis* (Second Edition), 1991, McGraw-Hill.

Locally convex spaces

Let $(X, \|\cdot\|_X)$ be a real normed space and let X^* be its dual, taken with respect to the norm of X. Then X^* is a normed space with $\|\Lambda\|_{X^*} := \sup\{|\Lambda x| : \ \|x\| \leq 1\}$ for $\Lambda \in X^*$. Recall that, since the field $\mathbb{R}$ is complete, X^* is a Banach space (Conway III.5.4). Let X^{**} be the dual of X^*, taken with respect to the operator norm of X^*. We equip X^{**} with the operator norm $\|T\|_{X^{**}} := \sup\{|T\Lambda| : \ \|T\|_{X^*} \leq 1\}$ for $T \in X^{**}$. Recall that the canonical embedding $J \colon X \to X^{**}$ is defined by $J(x)\Lambda := \Lambda x$ for $\Lambda \in \Lambda^*$. We have that J is an isometry, that is, $\|J(x)\|_{X^{**}} = \|x\|_X$ for all $x \in X$ (Conway III.6.7).

Convergences on normed spaces: Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence and $x \in X$. We say that $x_n \to x$ strongly if $\|x_n - x\|_X \to 0$ as $n \to \infty$. We say that $x_n \rightharpoonup x$ weakly if $\Lambda x_n \to \Lambda x$ as $n \to \infty$ for each $\Lambda \in X^*$. Let $\{\Lambda_n\}_{n\in\mathbb{N}} \subset X^*$ be a sequence and $\Lambda \in X^*$. We that $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ weakly* if $\Lambda_n x \to \Lambda x$ as $n \to \infty$ for all $x \in X$. Notice that, since X^* is a normed space, we can also consider the strong convergence with respect to $\|\cdot\|_{X^*}$ and the weak convergence induced by X^{**}.

Extremal points: Let X be a real vector space and let $K \subset X$ be a convex subset. We say that $a \in K$ is an *extremal point* of K if the following condition holds:

$$\text{if} \quad a = \lambda x_1 + (1-\lambda)x_2 \quad \text{for} \quad \lambda \in (0,1)\,,\ x_1, x_2 \in K \quad \text{then} \quad x_1 = x_2\,.$$

In other words, $a \in K$ is an extremal point if it does not lie in the interior of any open segment contained in K. We denote by $\mathrm{ext(K)}$ the set of extremal points of K. For an arbitrary set $E \subset X$ we define its *convex hull* by

$$\mathrm{co(E)} := \left\{ \sum_{j=1}^{n} \lambda_j x_j \ : \ n \in \mathbb{N}\,,\ x_1, \dots, x_n \in E\,,\ \sum_{j=1}^{n} \lambda_j = 1 \right\}\,,$$

that is, the set of all convex combinations of points of E.

Theorem (Krein-Milman): Let (X,τ) be a LCS. Assume that $K \subset X$ is non-empty, convex and compact with respect to τ. Then

$$\mathrm{ext(K)} \neq \emptyset \quad \text{and} \quad \mathrm{K} = \overline{\mathrm{co(ext(K))}}$$

where the closure is taken with respect to τ.

Problem 6.1. Let X be a real normed space and let $\Lambda_n, \Lambda \in X^*$ and $x_n, x \in X$ for $n \in \mathbb{N}$.

a) Show the following implications between convergences in X^*:

   i) if $\Lambda_n \to \Lambda$ strongly then $\Lambda_n \rightharpoonup \Lambda$ weakly,

   ii) if $\Lambda_n \rightharpoonup \Lambda$ weakly then $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ weakly*.

   Hint: Use the canonical embedding J.

b) Assume in addition that X is reflexive. Prove that in X^* we have that $\Lambda_n \rightharpoonup \Lambda$ weakly if and only if $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ weakly*.

   Hint: In this case, by definition, the canonical embedding J is surjective.

c) Prove that weak* limits in X^* and weak limits in X are unique.

   Hint: Use one of the corollaries of the Hahn-Banach Theorem (Conway III.6.8, Pag 79).

d) Assume that $x_n \rightharpoonup x$ weakly in X. Show that x_n is norm bounded, that is, $\sup_n \|x_n\|_X < \infty$.
   Hint: Use J and the Principle of Uniform Boundedness (Conway III.14.1, Pag 95).

e) Assume in addition that X is a Banach space. Show that if $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$ then $\sup_n \|\Lambda_n\|_{X^*} < \infty$.
   Hint: Use the Principle of Uniform Boundedness.

Problem 6.2 (15 pts).

a) Let X be a real normed space and $K := \{x \in X : \|x\| \le 1\}$ its (convex) unit ball. Show that

$$\mathrm{ext(K)} \subset \{\mathrm{x} \in \mathrm{X} : \|\mathrm{x}\| = 1\}\,.$$

b) Let $X = L^1(0,1)$, $K := \{f \in L^1(0,1) : \|f\|_1 \le 1\}$, where $\|f\|_1 := \int_0^1 |f(x)|\,dx$. Prove that

$$\mathrm{ext(K)} = \emptyset\,.$$

   Hint: If $f \in L^1(0,1)$ the function $\Psi\colon [0,1] \to \mathbb{R}$ defined by $\Psi(x) := \int_0^x |f(t)|\,dt$ is continuous and non-decreasing.

c) By using point (b) and Krein-Milman, prove that $L^1(0,1)$ is not the dual of a Banach space.
   Hint: Banach-Alaoglu (Conway V.3.1, Pag 130).

Distributions

Let $d \in \mathbb{N}, d \ge 1$ and $\Omega \subset \mathbb{R}^d$ be open and non-empty. We denote by $\mathcal{D}(\Omega)$ the set of $C^\infty(\Omega)$ functions with compact support. The space $\mathcal{D}(\Omega)$ is endowed with a topology τ which makes it into a complete LCS (Rudin 6.2). If $\phi_n, \varphi \in \mathcal{D}(\Omega)$, we have that $\phi_n \to \phi$ with respect to τ if and only if there exists a compact set $K \subset \Omega$ such that $\operatorname{supp}\phi_n \subset K$ for all n and

$$D^\alpha \phi_n \to D^\alpha \phi \quad \text{uniformly on } K \text{ for all } \alpha \in \mathbb{N}_0^d\,,$$

where $D^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$ (Rudin 6.5). We recall that the order of ∂^α is $|\alpha| := \sum_{i=1}^d \alpha_i$. The linear differential operator $D^\alpha\colon \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is continuous (Rudin 6.6). The space of distributions over Ω is denoted by $\mathcal{D}(\Omega)^*$ and it is defined as the set of linear operators $\Lambda\colon \mathcal{D}(\Omega) \to \mathbb{R}$ which are τ-continuous. For a linear operator $\Lambda\colon \mathcal{D}(\Omega) \to \mathbb{R}$ the following conditions are equivalent:

i) $\Lambda \in \mathcal{D}(\Omega)^*$,

ii) $\Lambda\phi_n \to 0$ whenever $\phi_n \to 0$ in $\mathcal{D}(\Omega)$,

iii) For each compact set $K \subset \Omega$ there exist constants $N \in \mathbb{N}_0$, $C > 0$ such that

$$|\Lambda\phi| \le C\,\|\phi\|_{K,N} \qquad \text{for all} \qquad \phi \in \mathcal{D}_K\,, \tag{1}$$

   where $D_K := \{\phi \in C^\infty(\Omega) : \operatorname{supp}\phi \subset K\}$ and

$$\|\phi\|_{K,N} := \max\{|\partial^\alpha \phi(x)| : x \in K,\, \alpha \in \mathbb{N}_0^d,\, |\alpha| \le N\}\,,$$

(Rudin 6.6 and 6.8). If the constant N in (1) is independent on K, we call the smallest N with such property the *order* of Λ.

For $f \in L^1_{\text{loc}}(\Omega)$ we define the distribution Λ_f via

$$\Lambda_f(\phi) := \int_\Omega f(x)\,\phi(x)\,dx \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega)\,.$$

Also, for $p \in \Omega$ define the delta distribution at p by $\delta_p(\phi) := \phi(p)$ for each $\phi \in \mathcal{D}(\Omega)$.

Derivatives: If $\Lambda \in \mathcal{D}(\Omega)^*, \alpha \in \mathbb{N}_0^d$ the α-derivative of Λ is the linear operator $\partial^\alpha \Lambda \colon \mathcal{D}(\Omega) \to \mathbb{R}$ defined by

$$(\partial^\alpha \Lambda)(\phi) := (-1)^{|\alpha|} \Lambda(\partial^\alpha \phi) \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega)\,.$$

We have that $\partial^\alpha \Lambda \in \mathcal{D}(\Omega)^*$ (Rudin 6.12). Notice that $\partial^\alpha \Lambda_f = \Lambda_{\partial^\alpha f}$ whenever f is regular and $f, \partial^\alpha f \in L^1_{\text{loc}}(\Omega)$.

Supports: Let $\Lambda \in \mathcal{D}(\Omega)^*$. If $\omega \subset \Omega$ is an open set, we say that $\Lambda = 0$ in ω if $\Lambda \phi = 0$ for all $\phi \in \mathcal{D}(\omega)$. We define W as the union of all the open sets $\omega \subset \Omega$ such that $\Lambda = 0$ in ω. The support of Λ is then defined by

$$\operatorname{supp} \Lambda := \Omega \smallsetminus W\,.$$

If $\operatorname{supp} \Lambda$ is compact, then Λ has finite order and it extends in a unique way to a linear continuous functional on $C^\infty(\Omega)$ (Rudin 6.24). Also recall the following structure theorem (Rudin 6.25):

Theorem 1. Let $\Lambda \in \mathcal{D}(\Omega)^*$ be such that $\operatorname{supp} \Lambda \subset \{p\}$ for some $p \in \Omega$. Then there exist $N \in \mathbb{N}$ and coefficients $c_\alpha \in \mathbb{R}$ for each $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq N$ such that

$$\Lambda = \sum_{|\alpha| \leq N} c_\alpha\, \partial^\alpha \delta_p \quad \text{in} \;\; \mathcal{D}(\Omega)^*\,.$$

Multiplication: Let $f \in C^\infty(\Omega)$ and $\Lambda \in \mathcal{D}(\Omega)^*$. Their multiplication is the distribution $f\Lambda$ defined by

$$(f\Lambda)(\phi) := \Lambda(f\phi) \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega)\,.$$

Limits: For $\Lambda_n \in \mathcal{D}(\Omega)^*$ we say that $\Lambda_n \to \Lambda$ in the sense of distributions if for each $\phi \in \mathcal{D}(\Omega)$ the limit

$$\lim_n \Lambda_n \phi = \Lambda \phi \tag{2}$$

exists and is finite. Whenever (2) is satisfied for some linear functional Λ, then automatically $\Lambda \in \mathcal{D}(\Omega)^*$ (Rudin 6.17). If $f_n \in L^1_{\text{loc}}(\Omega), \Lambda \in \mathcal{D}(\Omega)^*$, we write $f_n \to \Lambda$ in place of $\Lambda_{f_n} \to \Lambda$.

Convolutions 1: Let $\Lambda \in \mathcal{D}(\mathbb{R}^d)^*$, $\phi \in \mathcal{D}(\mathbb{R}^d)$. Introduce the translation and reflection operators

$$(\tau_x \phi)(y) := f(y-x)\,, \quad \check{\phi}(y) := \phi(-y) \quad \text{for} \quad y \in \mathbb{R}^d\,,$$

where $x \in \mathbb{R}^d$ is fixed. The *convolution* between Λ and ϕ is the map $\Lambda \star \phi \colon \mathbb{R}^d \to \mathbb{R}$ defined by

$$(\Lambda \star \phi)(x) := \Lambda(\tau_x \check{\phi}) \quad \text{for all} \;\; x \in \mathbb{R}^d\,. \tag{3}$$

It is important that ϕ is compactly supported for the definition to make sense. Notice that, if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\Lambda = \Lambda_f$, then (5) coincides with the classical definition, since

$$(\Lambda_f \star \phi)(x) = \int_{\mathbb{R}^d} f(y)(\tau_x \check{\phi})(y)\,dy = \int_{\mathbb{R}^d} f(y)\phi(x-y)\,dy = (f \star \phi)(x)$$

Motivated by the equality

$$\int_{R^d} (\tau_x f)(y)\phi(y)\,dy = \int_{R^d} f(y)(\tau_{-x}\phi)(y)\,dy\,,$$

we also define the translation of Λ by $x \in \mathbb{R}^d$ as the linear functional $\tau_x \Lambda \colon \mathcal{D}(\mathbb{R}^d) \to \mathbb{R}$ such that

$$(\tau_x \Lambda)(\phi) := \Lambda(\tau_{-x}\phi) \quad \text{for all} \;\; \phi \in \mathcal{D}(\mathbb{R}^d)\,. \tag{4}$$

Convolutions 2: Assume that $\Lambda \in \mathcal{D}(\mathbb{R}^d)^*$ has compact support and that $\phi \in C^\infty(\Omega)$. Since Λ extends to a linear continuous functional on $C^\infty(\Omega)$, it makes sense to define $\Lambda \star \phi \colon \mathbb{R}^d \to \mathbb{R}$ by

$$(\Lambda \star \phi)(x) := \Lambda(\tau_x \check{\phi}) \quad \text{for all} \;\; x \in \mathbb{R}^d\,, \tag{5}$$

in the same way we did in (5).

Problem 6.3. Let $\Lambda \in \mathcal{D}(\mathbb{R}^d)^*$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$.

a) Show that $\tau_x \Lambda$ belongs to $\mathcal{D}(\mathbb{R}^d)^*$ for all $x \in \mathbb{R}^d$, where τ_x is defined at (4).

b) Prove that
$$\tau_x(\Lambda \star \phi) = (\tau_x \Lambda) \star \phi = \Lambda \star (\tau_x \phi)$$
for all $x \in \mathbb{R}^d$.

c) Show that $\Lambda \star \phi \in C^\infty(\mathbb{R}^d)$ and that for each $\alpha \in \mathbb{N}_0^d$
$$D^\alpha(\Lambda \star \phi) = (D^\alpha \Lambda) \star \phi = \Lambda \star (D^\alpha \phi)\,.$$

d) Let $\rho_n \in \mathcal{D}(\mathbb{R}^d)$ be a sequence of mollifiers, that is, $\rho_n \geq 0$, $\operatorname{supp} \rho_n \subset B_{1/n}(0)$, $\int_{\mathbb{R}^d} \rho_n \, dx = 1$. Prove that
$$\lim_n \Lambda \star \rho_n = \Lambda \quad \text{ in } \ \mathcal{D}(\mathbb{R}^d)^*\,.$$
Hint: Use that $\Lambda \star (\phi \star \psi) = (\Lambda \star \phi) \star \psi$ for all $\Lambda \in \mathcal{D}(\mathbb{R}^d)^*$, $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$.

e) Assume in addition that Λ is compactly supported and let $\psi \in C^\infty(\mathbb{R}^d)$. Prove that the claims in points (b) and (c) hold for $\Lambda \star \psi$.

f) Assume in addition that Λ is compactly supported. Prove that $\Lambda \star \phi \in \mathcal{D}(\mathbb{R}^d)$.

Problem 6.4 (15 pts).

a) Let $\delta \in \mathcal{D}(\mathbb{R})^*$ be the Dirac distribution at 0, that is, $\delta(\phi) := \phi(0)$ for all $\phi \in \mathcal{D}(\mathbb{R})$. For each $m \in \mathbb{N}$, $m \geq 0$, characterize the set
$$A_m = \{ f \in C^\infty(\mathbb{R}) : \ f \delta^{(m)} = 0 \text{ in } \mathcal{D}(\mathbb{R})^* \}\,.$$

b) Give an example of $f \in C^\infty(\mathbb{R})$ and $\Lambda \in \mathcal{D}(\mathbb{R})^*$ such that $f = 0$ on $\operatorname{supp} \Lambda$ but $f\Lambda \neq 0$.

c) Fix $m \in \mathbb{N}$, $m \geq 1$. Show that they are equivalent:

   i) $x^m \Lambda = 0$ in $\mathcal{D}(\mathbb{R})^*$,

   ii) There exist $c_0, c_1, \dots, c_{m-1} \in \mathbb{R}$ such that $\Lambda = \sum_{k=0}^{m-1} c_k \delta^{(k)}$.

   Hint: If $x^m \Lambda = 0$, first prove that $\operatorname{supp} \Lambda \subset \{0\}$. Then you can use Theorem 1. It is useful to notice that, since Λ is compactly supported, you can test against functions in $C^\infty(\mathbb{R})$.

Problem 6.5 (20 pts). Suppose that $f \in L^1((-\infty, -\varepsilon) \cup (\varepsilon, \infty))$ for all $\varepsilon > 0$. The principal value integral of f is defined by
$$\mathrm{PV} \int_{\mathbb{R}} f(x)\, dx := \lim_{\varepsilon \to 0} \int_{\{|x| \geq \varepsilon\}} f(x)\, dx$$
whenever the limits exists (finite). Here $\{|x| \geq \varepsilon\}$ is a shorthand for $\{x \in \mathbb{R} : \ x \geq \varepsilon \ \text{ or } \ x \leq -\varepsilon\}$. For $\phi \in \mathcal{D}(\mathbb{R})$ define
$$\left(\mathrm{PV}\, \frac{1}{x} \right) (\phi) := \mathrm{PV} \int_{\mathbb{R}} \frac{\phi(x)}{x}\, dx\,.$$

a) Prove that $\mathrm{PV}\, \frac{1}{x}$ is well defined, that it belongs to $\mathcal{D}(\mathbb{R})^*$ and that its order is at most 1.

   Hint: Notice that $1/x$ is anti-symmetric, therefore $\int_{\{|x| \geq \varepsilon\}} x^{-1}\, dx = 0$.

b) Prove that $\mathrm{PV}\, \frac{1}{x}$ is a distribution of order 1.

   Hint: We already know that the order is at most 1. Assume by contradiction that the order is 0, so that for any $K \subset \mathbb{R}$ compact there exists $C > 0$ such that $|\left(\mathrm{PV}\, \frac{1}{x}\right)(\phi)| \leq C \, \|\phi\|_{K,0}$ for all $\phi \in \mathcal{D}_K$. Take $K = [0,1]$ and produce a sequence $\phi_n \in \mathcal{D}_K$ such such that $0 \leq \phi_n \leq 1$, which makes the previous estimate fail.

c) Show that, in the sense of distributions,

$$(\log|x|)' = \mathrm{PV}\,\frac{1}{x}\,.$$

d) Show that for all $\phi \in \mathcal{D}(\mathbb{R})$

$$\left(\mathrm{PV}\,\frac{1}{x}\right)'(\phi) = -\lim_{\varepsilon\to 0}\int_{\{|x|\geq\varepsilon\}} \frac{\phi(x)-\phi(0)}{x^2}\,dx\,.$$

Compact operators and spectral theory

For a Banach space X we denote by B_X its unit ball, that is, $B_X := \{x \in X\colon\ \|x\|_X \leq 1\}$. If Y is another Banach space, we denote by $\mathcal{L}(X,Y)$ the space of linear continuous operators $T\colon X \to Y$. Recall that $\mathcal{L}(X,Y)$ is a Banach space with the operator norm. We also denote $\mathcal{L}(X) := \mathcal{L}(X,X)$.

Compact operators: Let X,Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. We say that T is a *compact operator* if the closure of $T(B_X)$ is compact in Y. We denote the space of compact operators from X to Y by $\mathcal{K}(X,Y)$. Also we denote $\mathcal{K}(X) := \mathcal{K}(X,X)$.

Finite rank: Let X,Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. We say that T has *finite rank* if $T(X)$ is finite dimensional.

Adjoint: Let X,Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. The *adjoint* of T is the linear operator $T^*\colon Y^* \to X^*$ defined by

$$\langle T^*y^*, x\rangle_{X^*,X} = \langle y^*, Tx\rangle_{Y^*,Y} \quad \text{for all} \quad x \in X,\ y^* \in Y^*\,.$$

It is well-known that $T^* \in \mathcal{L}(Y^*,X^*)$, with $\|T\| = \|T^*\|$.

Theorem 2 (Brezis 6.1, 6.4): Let X,Y,Z be Banach spaces, $T \in \mathcal{L}(X,Y), S \in \mathcal{L}(Y,Z)$. Then:

i) $T \in \mathcal{K}(X,Y)$ if and only if $T^* \in \mathcal{K}(Y^*,X^*)$,

ii) If T has finite rank, then $T \in \mathcal{K}(X,Y)$,

iii) $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$: if $T_n \in \mathcal{K}(X,Y)$, $\|T_n - T\| \to 0$ then $T \in \mathcal{K}(X,Y)$,

iv) If $T \in \mathcal{K}(X,Y)$ or $S \in \mathcal{K}(X,Y)$, then $ST \in \mathcal{K}(X,Z)$.

Theorem 3 (Riesz's Lemma, Brezis 6.1): Let X be a normed space, and $M \subset X$ a closed subspace with $M \neq X$. Then for each $\varepsilon > 0$ there exists $x \in X$ such that $\|x\| = 1$ and $\mathrm{dist}(x,M) \geq 1-\varepsilon$.

Spectral theory: Let X be a Banach space, $T \in \mathcal{L}(X)$. The *resolvent* set of T is defined by

$$\rho(T) := \{\lambda \in \mathbb{R} :\ T - \lambda I \text{ is bijective from } X \text{ onto } X\}\,,$$

where I denotes the identity operator from X into itself. The *spectrum* of T is

$$\sigma(T) := \mathbb{R} \setminus \sigma(T)\,.$$

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of T if $\ker(T - \lambda I) \neq \{0\}$. We denote by $\mathrm{EV}(T)$ the set of eigenvalues of T. For $\lambda \in \mathrm{EV}(T)$, the corresponding *eigenspace* is $\ker(T - \lambda I)$. Notice that $\mathrm{EV}(T) \subset \sigma(T)$, but they are not equal in general.

Theorem 4 (Brezis 6.7, 6.8): Let X be a Banach space and $T \in \mathcal{L}(X)$. Then $\sigma(T)$ is a compact set, $\sigma(T) = \sigma(T^*)$ and

$$\sigma(T) \subset [-\|T\|, \|T\|]\,.$$

Assume in addition that X is infinite dimensional and $T \in \mathcal{K}(X)$. Then

i) $0 \in \sigma(T)$,

ii) $\sigma(T) \setminus \{0\} = \mathrm{EV}(T) \setminus \{0\}$,

iii) Either $\sigma(T) = \{0\}$, or $\sigma(T) \setminus \{0\}$ is a finite set, or $\sigma(T) \setminus \{0\} = \{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_n \to 0$.

Relative compactness in $C(X)$: Assume that (X, d) is a compact metric space. We denote by $C(X)$ the space of continuous functions $f\colon X \to \mathbb{R}$. Then $C(X)$ is a Banach space with the supremum norm $\|f\|_\infty := \sup_{x\in X} |f(x)|$. For a family $\mathcal{A} \subset C(X)$ we say that $\mathcal{A}$ is *uniformly bounded* if there exists a constant $M > 0$ such that

$$\sup_{x\in X} |f(x)| \leq M \quad \text{for all} \quad f \in \mathcal{A}\,.$$

We say that $\mathcal{A}$ is *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ (depending only on ε) with the following property:

$$\text{for all } \; x, y \in X \; \text{ such that } \; d(x, y) < \delta\,, \; \text{ it follows that } \; |f(x) - f(y)| < \varepsilon \; \text{ for all } \; f \in \mathcal{A}\,.$$

A characterization of relative compactness in $C(X)$ is given by the following:

Theorem 5 (Ascoli-Arzelà): Let (X, d) be a compact metric space. Let $\mathcal{A} \subset C(X)$. They are equivalent:

i) The closure of $\mathcal{A}$ is compact in $C(X)$ (with respect to the supremum norm);

ii) $\mathcal{A}$ is uniformly bounded and equicontinuous;

iii) each sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{A}$ admits a subsequence converging uniformly.

Relative compactness in L^p: Let $d \in \mathbb{N}, d \geq 1$. For a map $f\colon \mathbb{R}^d \to \mathbb{R}$ we define its translation by $h \in \mathbb{R}^d$ as the new map $\tau_h f\colon \mathbb{R}^d \to \mathbb{R}$ defined by $(\tau_h f)(x) := (x - h)$. The following theorem is a version of Ascoli-Arzelà for L^p spaces.

Theorem 6 (Fréchet-Kolmogorov): Let $1 \leq p < \infty$ and consider a family $\mathcal{A} \subset L^p(\mathbb{R}^d)$. Suppose that $\mathcal{A}$ is bounded, that is,

$$\sup_{f\in\mathcal{A}} \|f\|_{L^p(\mathbb{R}^d)} < +\infty\,.$$

Moreover assume that

$$\lim_{|h|\to 0} \|\tau_h f - f\|_{L^p(\mathbb{R}^d)} = 0 \quad \text{uniformly in} \;\; f \in \mathcal{A}$$

that is, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\tau_h f - f\|_{L^p} < \varepsilon$ for all $f \in \mathcal{A}$, $h \in \mathbb{R}^d$ with $|h| < \delta$. Then the closure of $\mathcal{A}|_\Omega$ in $L^p(\Omega)$ is compact for any Lebesugue measurable set $\Omega \subset \mathbb{R}^d$ with $|\Omega| < \infty$.

In the above theorem we denote by $\mathcal{A}|_\Omega$ the restriction to Ω of functions in $\mathcal{A}$.

Problem 6.6 (10 pts).

a) Let X be a normed space. Show that the identity map $I\colon X \to X$ is compact if and only if $\dim X < +\infty$.

   Hint: Riesz Lemma (Theorem 3).

b) Consider $C^1[0, 1]$ equipped with the norm $\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$ and $C[0, 1]$ equipped with the supremum norm. Prove that the identity $I\colon C^1[0, 1] \to C[0, 1]$ is continuous and compact.

   Hint: Use Theorem 5.

Problem 6.7 (10 pts). Let H be a real Hilbert space and $T \in \mathcal{L}(H)$. Let $x_n, x \in H$ for $n \in \mathbb{N}$.

a) Show that $x_n \to x$ strongly in H if and only if

$$x_n \rightharpoonup x \;\text{ weakly in }\; H \;\text{ and }\; \|x_n\|_H \to \|x\|_H\,.$$

b) Show that $T \in \mathcal{K}(H)$ if and only if the following condition holds:

$$\text{If } \; x_n \rightharpoonup x \; \text{ weakly in } \; H\,, \; \text{ then } \; Tx_n \to Tx \; \text{ strongly in } \; H\,.$$

   Hint: Since H is reflexive, $T(B_H)$ is closed (see Problem 6.8 Point (c)).

Problem 6.8.

a) Let X, Y be normed spaces, $T \in \mathcal{L}(X, Y)$. Assume there exists a constant $c > 0$ such that

$$\|Tx\|_Y \geq c\,\|x\|_X \qquad \text{for all} \qquad x \in X\,.$$

Show that T is compact if and only if $\dim X < +\infty$.

b) Let X be a Banach space with $\dim X = +\infty$ and let $T \in \mathcal{K}(X)$. Show that T cannot be surjective, that is, there exists $y \in X$ such that the equation

$$Tx = y$$

has no solution in X.

c) Let X, Y be Banach spaces and assume that X is reflexive. Let $T \in \mathcal{L}(X, Y)$ and $M \subset X$ be closed, convex and bounded.

   i) Show that $T(M)$ is closed in Y.

   ii) In addition, assume that $T \in \mathcal{K}(X, Y)$. Show that $T(M)$ is compact.

d) Let H be a Hilbert space and $T \in \mathcal{K}(H)$. Show that T attains its norm, that is, there exists $\hat{x} \in H$ such that $\|\hat{x}\| \leq 1$ and $\|T\| = \|T\hat{x}\|$.

Problem 6.9 (30 pts). Consider the space $C[0,1]$ equipped with the supremum norm and let $1 \leq p \leq \infty$. Define the linear operator $T\colon L^p(0,1) \to L^p(0,1)$ by

$$(Tf)(x) := \int_0^x f(t)\,dt \qquad \text{for} \qquad x \in [0,1]\,.$$

Also consider the linear operator $S\colon C[0,1] \to C[0,1]$ defined by $Sf := Tf$ for $f \in C[0,1]$.

a) Prove that S is bounded and compute $\|S\|$.

b) Let $B := \{f \in C[0,1] : \|f\|_\infty \leq 1\}$. Prove that $S(B)$ is not closed.

   Hint: Notice that $Sf \in C^1[0,1]$ for all $f \in C[0,1]$. Therefore construct a sequence $f_n \in B$ such that $Sf_n \to g$ uniformly but $g \notin C^1[0,1]$.

c) Prove that S is compact.

d) Prove that T is bounded for all $p \in [1,\infty]$ and compute its adjoint T^*.

e) Prove that T is compact for each $p \in [1,\infty]$.

   Hint: For $1 < p \leq \infty$ use the fact that $Tf \in C[0,1]$. Therefore if you show compactness in $C[0,1]$ (by employing Theorem 5), you also have it in $L^p(0,1)$. For $p = 1$ you do not have compactness in $C[0,1]$ (see point (g)), but you can still prove compactness in $L^1(0,1)$ by means of Theorem 6.

f) Compute $\sigma(T)$, $\mathrm{EV}(T)$ and $\rho(T)$.

   Hint: Try to compute $\mathrm{EV}(T)$ first. Remember that if $f \in L^p(0,1)$, then its primitive is Sobolev (see Problem 2.3). By a bootstrap argument you can infer regularity of the eigenvectors.

g) Show that $T\colon L^1(0,1) \to C[0,1]$ is not compact.

   Hint: Consider $f_n(x) := n\chi_{(0,1/n)}(x)$.