MAT.405 / WS 2019/2020
Silvio Fanzon

Institute for Mathematics
and Scientific Computing

Advanced Functional Analysis

Problem Sheet 5

Due date: December 9, 2019

In the following we will adopt the definitions given in Worksheet 4.

Weak topologies: let X be a real LCS and denote by τ the topology induced by the family of seminorms $\mathcal{P}$. Consider the topological dual of X with respect to τ, defined by

$$X^* = (X, \tau)^* := \{x^* \colon X \to \mathbb{R} : \ x^* \ \text{ linear and } \tau\text{-continuous}\} .$$

Notice that X^* has a natural vector space structure, with

$$(x^* + \lambda y^*)(x) := x^*(x) + \lambda y^*(x) \quad \text{ for } \quad x^*, y^* \in X^*, \lambda \in \mathbb{R}, x \in X$$

For some $x^* \in X^*$ we define the map

$$p_{x^*} \colon \ X \to \mathbb{R} \quad \text{ by } \quad p_{x^*}(x) := |x^*(x)| \quad \text{ for all } \quad x \in X .$$

It is immediate to check that p_{x^*} is a seminorm on X. The *weak topology* on X (denoted by wk) is defined as the topology induced by the family of seminorms $\{p_{x^*} \colon \ x^* \in X^*\}$. Similarly, for $x \in X$ define the seminorm

$$p_x \colon X^* \to \mathbb{R} \quad \text{ by } \quad p_x(x^*) := |x^*(x)| \quad \text{ for all } \quad x^* \in X^* .$$

The *weak* topology* on X^* (denoted by wk^*) is the topology induced by the family of seminorms $\{p_x \colon \ x \in X\}$. Notice that both (X, wk) and (X^*, wk^*) are LCS. Therefore one has naturally a second topology on X, in addition to τ.

Problem 5.1 (30 pts).

a) Let X be a real vector space and $\varphi, \varphi_1, \dots, \varphi_n \colon X \to \mathbb{R}$ be linear functionals. Show that (i) and (ii) are equivalent, where

   i) There exist $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $\varphi(x) = \sum_{j=1}^n \alpha_j \varphi_j(x)$ for all $x \in X$,

   ii) $(\cap_{j=1}^n \ker \varphi_j) \subset \ker \varphi$, where ker denotes the kernel.

   Hint: Consider the maps $\pi \colon X \to \mathbb{R}^n$ where $\pi(x) := (\varphi_1(x), \dots, \varphi_n(x))$ and $F \colon \pi(X) \to \mathbb{R}$ where $F(\pi(x)) := \varphi(x)$. Is F well defined?

b) Let X be a real LCS, τ its topology and X^* its dual. Prove that

$$(X^*, wk^*)^* = X ,$$

   in the sense that if $\varphi \colon X^* \to \mathbb{R}$ is linear and weak* continuous, then there exists $x \in X$ (depending on φ) such that $\varphi(x^*) = x^*(x)$ for all $x^* \in X^*$.

   Hint: Recall the definition of open set with respect to a topology defined by seminorms (see Worksheet 4) and apply it to $\{x^* \in X^* \colon |\varphi(x^*)| < 1\}$. Then use (a).

Metrizable LCS: Let (X, τ) be a real LCS with the topology τ induced by the separating family of seminorms $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$ (so in particular (X, τ) is Hausdorff). Assume that $d\colon X \times X \to \mathbb{R}$ is a translation invariant metric on X, that is, d is a metric on X and

$$d(x + a, y + a) = d(x, y) \quad \text{for all} \quad x, y, a \in X\,. \tag{1}$$

Denote by τ_d the topology induced by d on X. Since (1) holds, the topology τ_d is local, meaning that it is determined (up to translations) by the neighbourhood system at 0, given by $\mathcal{U} = \{U_\delta \colon \delta > 0\}$ where $U_\delta := \{x \in X : d(x, 0) < \delta\}$. Recall that also τ is local, with the neighbourhoods of 0 given by $\mathcal{V} = \{V_{\alpha,\varepsilon} \colon \alpha \in A, \varepsilon > 0\}$ where $V_{\alpha,\varepsilon} := \{x \in X \colon p_\alpha(x) < \varepsilon\}$.

We say that X is *metrized* by d if $\tau = \tau_d$, in the sense that for each $U_\delta \in \mathcal{U}$ there exists some $V_{\alpha,\varepsilon} \in \mathcal{V}$ such that $V_{\alpha,\varepsilon} \subset U_\delta$, and for each $V_{\alpha,\varepsilon} \in \mathcal{V}$ there exists some $U_\delta \in \mathcal{U}$ such that $U_\delta \subset V_{\alpha,\varepsilon}$. We say that (X, τ) is *metrizable* if there exists a translation invariant metric d such that $\tau = \tau_d$.

Problem 5.2 (30 pts). Let X be a real LCS whose topology τ is generated by a countable family of separating seminorms $\mathcal{P} := \{p_n\}_{n \in \mathbb{N}}$. Show that X is metrizable, by following the strategy below:

a) Define the map $d\colon X \times X \to \mathbb{R}$ by

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \, \frac{p_n(x - y)}{1 + p_n(x - y)}\,.$$

Show that d is a translation invariant metric on X.

Hint: The scalar map $t \mapsto \frac{t}{1+t}$ is increasing for $t > -1$ and bounded by 1 from above.

b) Denote by τ_d the topology induced by d on X. Show that $\tau_d = \tau$ (in particular you showed that (X, τ) is metrizable).

Hint: You can assume that $p_n(x) \leq p_{n+1}(x)$ for all $n \in \mathbb{N}$, $x \in X$ (since the family of seminorms $\{q_n\}$ defined by $q_n(x) := \max_{k \leq n} p_k(x)$ induces the topology τ and $q_n \leq q_{n+1}$). Also recall that $\sum_{n=1}^{k} 2^{-n} = 1 - 2^{-k}$ and $\sum_{n=k+1}^{\infty} 2^{-n} = 2^{-k}$.

Dual of continuous functions: Let (K, τ) be a compact Hausdorff topological space and consider the Banach space $C(K) := \{f\colon K \to \mathbb{R} : f \text{ is } \tau\text{-continuous}\}$ equipped with the supremum norm $\|f\|_\infty := \sup_{x \in K} |f(x)|$. We denote by $\mathcal{M}(K)$ the space of bounded Borel measures on K. If $\mu \in \mathcal{M}(K)$ we denote by $|\mu|$ its total variation measure. Recall that $\mathcal{M}(K)$ equipped with the norm $\|\mu\| := |\mu|(K)$ is a Banach space. The Riesz theorem states that the dual of $(C(K), \|\cdot\|_\infty)$ coincides with $\mathcal{M}(K)$. More precisely:

Theorem (Riesz): Let (K, τ) be a compact Hausdorff topological space. Let $\mu \in \mathcal{M}(K)$ and define the functional $\Lambda_\mu \colon C(K) \to \mathbb{R}$ by

$$\Lambda_\mu(f) := \int_K \varphi(x)\, d\mu(x) \quad \text{for all} \quad f \in C(K)\,.$$

Then Λ_μ is linear and continuous, and its operator norm satisfies $\|\Lambda_\mu\| = |\mu|(K)$. Conversely, let $\Lambda\colon C(K) \to \mathbb{R}$ be a linear and continuous functional. Then there exists a unique $\mu \in \mathcal{M}(K)$ such that $\Lambda = \Lambda_\mu$.

In view of the above theorem we have that a sequence $\{f_n\}_{n \in \mathbb{N}} \subset C(K)$ is *weakly* converging to some $f \in C(K)$ if and only if

$$\int_K f_n(x)\, d\mu(x) \to \int_K f(x)\, d\mu(x) \quad \text{as} \quad n \to \infty$$

for all $\mu \in \mathcal{M}(K)$ fixed. We also recall the dominated convergence theorem in higher generality:

Theorem (Dominated convergence): Let K be a compact Hausdorff space and fix $\mu \in \mathcal{M}(K)$. Let $f_n \colon K \to \mathbb{R}$ be a sequence of μ-measurable maps such that

i) $f_n(x) \to f(x)$ for μ-a.e. x in K, as $n \to \infty$,

ii) $\sup_n |f_n(x)| \leq g(x)$ for μ-a.e. x in K, with $g\colon K \to \mathbb{R}$ is μ-measurable and $\int_K g(x)\, d\mu(x) < \infty$.

Then f is μ-measurable and $\int_K |f_n(x) - f(x)|\, d\mu(x) \to 0$ as $n \to \infty$.

Problem 5.3 (20 pts). Let K be a compact Hausdorff space and consider the Banach space $C(K)$ equipped with the supremum norm.

a) Let $f_n, f \in C(K)$ for $n \in \mathbb{N}$. Show that conditions (i) and (ii) given below are equivalent:

   i) f_n weakly converges to f as $n \to \infty$,

   ii) $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in K$ fixed.

   Hint: Use dominated convergence and Riesz theorem.

b) Let $\varphi \in C[0,1]$ be fixed and define the sequence $\{f_n\} \subset C[0,1]$ by setting

$$f_n(x) := \varphi(x^n) \quad \text{for} \quad x \in [0,1] .$$

   By using (a), prove that conditions (i') and (ii') given below are equivalent:

   i') f_n weakly converges to some $f \in C[0,1]$ as $n \to \infty$,

   ii') $\varphi(0) = \varphi(1)$.

Weak completeness: Let X be a real normed space. We say that a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is *weakly Cauchy* if for every $x^* \in X^*$ the sequence $\{x^*(x_n)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. We say that X is *weakly sequentially complete* if every weak Cauchy sequence is weakly convergent.

In the next exercise we will see that reflexivity is sufficient for weak sequential completeness, but not necessary.

Problem 5.4 (20 pts).

a) Let X be a reflexive Banach space. Show that X is weakly sequentially complete.

   Hint: Principle of uniform boundedness and Banach-Alaoglu might be useful.

b) Consider the Banach space $C[0,1]$ equipped with the supremum norm. Prove that $C[0,1]$ is not weakly sequentially complete.

   Hint: Examine the sequence f_n defined by $f_n(x) := 1 - nx$ if $0 \le x \le 1/n$ and $f_n(x) := 0$ if $1/n \le x \le 1$.