Institute for Mathematics
and Scientific Computing

UNI GRAZ

Advanced Functional Analysis

Problem Sheet 4

Due date: November 25, 2019

The goal of Problems 4.1 and 4.2 is to prove Morrey's inequality and the consequent embedding.

Theorem 1 (Morrey's inequality): Let $N \in \mathbb{N}, N \geq 1$, $N < p \leq \infty$ and set $\alpha := 1 - \frac{N}{p}$. There exists a constant $C > 0$ depending only on N and p, such that

$$\|f\|_{C^{0,\alpha}(\mathbb{R}^N)} \leq C \|f\|_{W^{1,p}(\mathbb{R}^N)} \qquad \text{for all} \qquad f \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)\,. \tag{1}$$

Theorem 2 (Embedding): Let $N \in \mathbb{N}, N \geq 1$, $N < p \leq \infty$ and $\alpha := 1 - \frac{N}{p}$. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with C^1 boundary. Then every function $f \in W^{1,p}(\Omega)$ coincides a.e. in Ω with a function $\tilde{f} \in C^{0,\alpha}(\overline{\Omega})$. Moreover there exists a constant $C > 0$ such that

$$\left\|\tilde{f}\right\|_{C^{0,\alpha}(\overline{\Omega})} \leq C \|f\|_{W^{1,p}(\Omega)} \qquad \text{for all} \qquad f \in W^{1,p}(\Omega)\,.$$

Recall: For $0 < \alpha \leq 1$, $C^{0,\alpha}(\overline{\Omega})$ is the space of continuous bounded functions $f \colon \Omega \to \mathbb{R}$ normed by

$$\left\|\tilde{f}\right\|_{C^{0,\alpha}(\overline{\Omega})} := \sup_{x \in \overline{\Omega}} |f(x)| + \sup_{\substack{x,y \in \overline{\Omega} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}\,.$$

Problem 4.1 (30 pts). Let $N \in \mathbb{N}, N \geq 1$, $N < p \leq \infty$ and $\alpha := 1 - \frac{N}{p}$. Prove Theorem 1 by following the strategy below:

a) Let $B_r(x)$ be the open ball centred at $x \in \mathbb{R}^N$ with radius $r > 0$. Show that

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)|\, dy \leq C \int_{B_r(x)} \frac{|\nabla f(y)|}{|x-y|^{N-1}}\, dy \tag{2}$$

for all $f \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ and all balls $B_r(x) \subset \mathbb{R}^N$, where $C > 0$ is a constant.

Hint: Recall that $\int_{B_r(x)} f(y)\, dy = \int_0^r \int_{\partial B_\rho(x)} f(w)\, dS(w)\, d\rho$, where dS is the surface element on $\partial B_\rho(x)$ (or the $(N-1)$-dimensional Hausdorff measure restricted to $\partial B_\rho(x)$). Also $|B_r(x)| = \tilde{C} r^N$, where $\tilde{C} > 0$ depends only on N.

b) Prove, by using (2), that there exists $C > 0$ (depending only on N and p) such that

$$\sup_{x \in \mathbb{R}^N} |f(x)| \leq C \|f\|_{W^{1,p}(\mathbb{R}^N)} \qquad \text{for all} \qquad f \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)\,.$$

c) Prove that there exists $C > 0$ (depending only on N and p) such that

$$\sup_{\substack{x,y \in \mathbb{R}^N \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq C \|\nabla f\|_{L^p(\mathbb{R}^N)} \qquad \text{for all} \qquad f \in C^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)\,.$$

Hint: Fix $x, y \in \mathbb{R}^N$ and define $r := |x-y|$, $W := B_r(x) \cap B_r(y)$. Use (2) to estimate $\frac{1}{|W|} \int_W |f(x) - f(z)| + |f(y) - f(z)|\, dz$.

Problem 4.2 (10 pts). Prove the embedding Theorem 2.

Hint: you can use (1) in conjunction with extension and density arguments.

We now pass to some exercises on topological vector spaces and on locally convex spaces. Let us establish some definitions first.

Topological vector spaces: Let X be real vector space (that is, X is a vector space over $\mathbb{R}$). We say that X is a *topological vector space* (TVS) with the topology τ if the operations

$$(x, y) \mapsto x + y\,, \quad X \times X \to X\,,$$
$$(\lambda, x) \mapsto \lambda x\,, \quad \mathbb{R} \times X \to X\,,$$

are continuous with respect to τ.

Subsets: Let X be a real vector space and $V \subset X$ be a subset. We say that V is *convex* if for all $x, y \in V$, $\lambda \in [0, 1]$ we have that $\lambda x + (1 - \lambda)y \in V$. We say that V is *balanced* if $\alpha x \in V$ whenever $x \in V$ and $|\alpha| \leq 1$. Notice that a non-empty balanced set always contains the origin. Finally, we say that V is *absorbing* at the point $a \in V$ if for every $x \in X$ there exists $\varepsilon > 0$ such that $a + tx \in V$ for all $0 \leq t < \varepsilon$.

Seminorms: Let X be a real vector space. A *seminorm* on X is a function $p\colon X \to [0, \infty)$ (notice that p is required to be non-negative and finite) satisfying the following properties:

i) p is positively homogenous, that is, $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{R}$, $x \in X$.

ii) p is subadditive, that is, $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Notice that $p(0) = 0$ as a consequence of (i). Recall that a norm is a seminorm with the additional property that if $p(x) = 0$ for some $x \in X$, then necessarily $x = 0$.

Minkowski functional: Let X be a real vector space and $V \subset X$ be a subset. The Minkowski functional associated to V is the function $p = p_V : X \to [0, \infty]$ defined by

$$p(x) := \inf \{t \geq 0 : \ x \in tV\} \ ,$$

where we define $p(x) := \infty$ if the set $\{t \geq 0 : \ x \in tV\}$ is empty.

Families of seminorms: Let X be a real vector space and $\mathcal{P}$ be a family of seminorms on X. Let $\tau_{\mathcal{P}}$ be the topology that has a subbase the sets

$$\{x \in X : \ p(x - x_0) < \varepsilon\}\,,$$

for some $x_0 \in X$, $p \in \mathcal{P}$, $\varepsilon > 0$. In other words, $A \subset X$ is open (with respect to $\tau_{\mathcal{P}}$) if and only if for every $x_0 \in A$ there exist $p_1, \dots, p_n \in \mathcal{P}$, $\varepsilon_1, \dots, \varepsilon_n > 0$, $n \in \mathbb{N}$, such that

$$\bigcap_{j=1}^{n} \{x \in X : \ p_j(x - x_0) < \varepsilon_j\} \subset A\,.$$

In particular, $\tau_{\mathcal{P}}$ is the coarsest topology for which all the elements of $\mathcal{P}$ are continuous. It is standard that X equipped with $\tau_{\mathcal{P}}$ is a TVS. We say that $\mathcal{P}$ is *separating* if

$$\bigcap_{p \in \mathcal{P}} \{x \in X : \ p(x) = 0\} = \{0\}\,.$$

Locally convex spaces: Let (X, τ) be a TVS. We say that X is a *locally convex space* (LCS) if $\tau = \tau_{\mathcal{P}}$ for some family $\mathcal{P}$ of seminorms on X.

Some topological definitions: Let X be a topological space and fix $x \in X$. We say that $U \subset X$ is a neighbourhood of x if there exists $A \subset X$ open, such that $x \in A$ and $A \subset U$. We say that X is *Hausdorff* if for all $x, y \in X$ with $x \neq y$, there exist $U, V \subset X$ neighbourhoods of x and y

respectively, such that $U \cap V = \emptyset$. If Y is another topological space, we say that a map $f\colon X \to Y$ is an *homeomorphism* if f is continuous, invertible, and the inverse of f is continuous. Finally, we say that a collection of open sets $\mathcal{N}$ is a *basis for the neighbourhood system* of a point $x \in X$ if the following properties are satisfied: $x \in N$ for all $N \in \mathcal{N}$; if $A \subset X$ is an open set such that $x \in A$, then there exists some $N \in \mathcal{N}$ such that $N \subset A$.

Problem 4.3 (20 pts). Let X be a TVS. Denote by τ its topology.

a) Fix $x_0 \in X$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Define the translation and dilation maps $T, D\colon X \to X$ by $T(x) := x + x_0$, $D(x) := \lambda x$ for all $x \in X$. Show that T and D are homeomorphisms.

b) Let $A \subset X$ be open. Show that A is absorbing at each of its points.

   Hint: Recall that the vector space operations on X are continuous by definition.

c) Let $n \in \mathbb{N}$ and $p_1, \dots, p_n$ be seminorms on X. Define

$$p(x) := \max_{j=1,\dots,n} p_j(x) \quad \text{for all} \quad x \in X\,.$$

   Show that p is a seminorm on X.

d) Assume that X is a LCS, that is, $\tau = \tau_{\mathcal{P}}$ for some family of seminorms $\mathcal{P}$. Show that (X, τ) is Hausdorff if the family $\mathcal{P}$ is separating.

In the next exercise we show that the topology of a LCS X can be equivalently described by a system of open, convex, balanced sets.

Problem 4.4 (40 pts). Let X be a real vector space.

a) Let $p\colon X \to [0, \infty)$ be a seminorm on X and define the set

$$V := \{x \in X : \, p(x) < 1\}\,. \tag{3}$$

   Show that V is convex, balanced and absorbing at each of its points.

b) Let $V \subset X$ be a subset. Assume that V is convex, balanced and absorbing at each of its points. Prove that the Minkowski functional p associated to V is a seminorm on X. Moreover show that p and V satisfy relation (3).

   Hint: if V is convex and $\alpha, \beta \geq 0$, then $\alpha V + \beta V \subset (\alpha + \beta)V$ (prove it). Also if $p(x) < \infty$, by definition of infimum, for each $\varepsilon > 0$ there exists $t \geq 0$ such that $t \leq p(x) + \varepsilon$ and $x \in tV$.

c) Assume that X is a TVS and define

$$\mathcal{U} := \{V \subset X : \, V \ \text{ open, convex and balanced }\}\,.$$

   Show that X is a LCS if and only if $\mathcal{U}$ is a basis for the neighbourhood system at 0.

   Hint: points (a), (b) from this exercise and points (b), (c) from Problem 4.3 are very useful!