MAT.405 / WS 2019/2020
Silvio Fanzon

Institute for Mathematics
and Scientific Computing

Advanced Functional Analysis

Problem Sheet 3

Due date: November 11, 2019

I will refer to the book H.Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, 2011, Springer-Verlag New York, which you can download from the university library website.

Problem 3.1 (30 pts).

a) Let $f(x) := |x|$ for $x \in [-1,1]$. Show that $f \in W^{1,p}(-1,1)$ for all $1 \leq p \leq \infty$.

b) Let $f(x) := -\chi_{(-1,0)}(x) + \chi_{(0,1)}(x) = \operatorname{sign} x$. Show that $f \notin W^{1,p}(-1,1)$ for any $1 \leq p \leq \infty$, **without** using the characterization theorem for 1D Sobolev functions (Theorem 1 in Worksheet 2).

   Hint: assume by contradiction that f has a weak derivative in $L^1(-1,1)$. Then derive a contradiction by means of a sequence of functions localized around the jump, i.e. a sequence $\psi_n \in C_c^\infty(-1,1)$ such that $0 \leq \psi_j \leq 1$, $\psi_j(1) = 1$ and $\lim_j \psi_j(x) = 0$ for all $x \neq 0$.

Problem 3.2 (30 pts). Let $N \geq 3$. Assume that $\Omega \subset \mathbb{R}^N$ is an open connected set with regular boundary $\partial\Omega$. Assume that $A \colon \Omega \to \mathbb{R}^{N \times N}$ is a Lebesgue measurable matrix field such that there exist constants $C \geq 0, \alpha > 0$ satisfying

$$|A(x)| \leq C\,, \quad A(x)\xi \cdot \xi \geq \alpha |\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^N\,, \quad \text{for a.e. } x \in \Omega\,.$$

Let $K > 0$ be the Poincaré constant of Ω, that is, a constant such that

$$\|u\|^2_{L^2(\Omega)} \leq K\, \|\nabla u\|^2_{L^2(\Omega)} \qquad \text{for all} \qquad u \in H^1_0(\Omega)\,,$$

(see Corollary 9.19 Brezis). Fix $\lambda \in \mathbb{R}$ such that $\lambda + \alpha/K > 0$. Let $f \in L^{2_*}(\Omega)$, where 2_* is the Hölder conjugate of $2^* := \frac{2N}{N-2}$ (that is, $\frac{1}{2^*} + \frac{1}{2_*} = 1$). We say that $u \in H^1_0(\Omega)$ is a *weak solution* to the boundary value problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u(x)) + \lambda u(x) = f(x) & \text{if} \quad x \in \Omega\,, \\ u(x) = 0 & \text{if} \quad x \in \partial\Omega\,, \end{cases}$$

if u satisfies

$$\int_\Omega A(x)\nabla u(x) \cdot \nabla v(x)\, dx + \lambda \int_\Omega u(x)v(x)\, dx = \int_\Omega v(x) f(x)\, dx \qquad \text{for all} \qquad v \in H^1_0(\Omega)\,. \tag{1}$$

Show that there exists a unique solution $u \in H^1_0(\Omega)$ to (1).
Hint: consider the Hilbert space $H := H^1_0(\Omega)$ and apply Lax-Milgram (see Worksheet 1). It is also useful to recall Sobolev embeddings and Poincaré's inequality.

The goal of the next exercise is to prove the following theorem:

Theorem 1 (Partitions of unity): Let $N, k \in \mathbb{N}$ with $N, k \geq 1$. Let $\Omega \subset \mathbb{R}^N$ be a bounded set such that $\Omega \subset\subset \cup_{j=1}^k U_j$, where $U_j \subset \mathbb{R}^N$ is open for each $j = 1, \dots, k$. Then there exist functions $\eta_j \in C_c^\infty(U_j)$ such that $0 \leq \eta_j \leq 1$, $\operatorname{supp} \eta_j \subset U_j$ for all $j = 1, \dots, k$, and

$$\sum_{j=1}^k \eta_j(x) = 1 \quad \text{for all} \quad x \in \Omega\,.$$

We recall that a family $\{\eta_j\}_{j=1}^k$ satisfying the properties of Theorem 1 is called a *partition of unity* subordinate to the open cover $\{U_j\}_{j=1}^k$ of Ω.

Problem 3.3 (40 pts). Let $N, k \in \mathbb{N}$, $N, k \geq 1$.

a) Let $K \subset \mathbb{R}^N$ be compact and $U \subset \mathbb{R}^N$ be open, such that $K \subset U$. Show that there exists $\psi \in C_c^\infty(\mathbb{R}^N)$ with the following properties:

- $0 \leq \psi \leq 1$ in $\mathbb{R}^N$,
- $\psi(x) = 1$ for all $x \in K$,
- $\operatorname{supp} \psi \subset U$.

Hint: For $\varepsilon > 0$ consider $K_\varepsilon := K + \overline{B_\varepsilon} = \{x \in \mathbb{R}^N : \operatorname{dist}(x, K) \leq \varepsilon\}$. Take $\varepsilon > 0$ small enough so that $K_{3\varepsilon} \subset U$ (you can do it since $K \subset\subset U$). Then use the standard mollifiers ρ_ε to construct ψ (recall the basic properties of mollifiers: $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N)$, $\rho_\varepsilon \geq 0$, $\operatorname{supp} \rho_\varepsilon \subset \overline{B_\varepsilon}$, $\int_{\mathbb{R}^N} \rho_\varepsilon(x)\,dx = 1$). Also recall the properties of convolutions from Worksheet 2.

b) (Refining the cover) Let $K \subset \mathbb{R}^N$ be compact and $U_j \subset \mathbb{R}^N$ be open for all $j = 1, \dots, k$. Assume that $K \subset \cup_{j=1}^k U_j$. Show that exists a family $\{K_j\}_{j=1}^k$ of compact sets of $\mathbb{R}^N$ such that $K_j \subset U_j$ for all $j = 1, \dots, k$ and $K \subset \cup_{j=1}^k K_j$.

Hint: since K is bounded, it is not restrictive to assume that U_j is bounded for all $j = 1, \dots, k$. It might be useful to consider the sets $U_{n,j} := \{x \in U_j : \operatorname{dist}(x, \partial U_j) > 1/n\}$ for $n \in \mathbb{N}$. Also recall the topological definition of compactness: from an arbitrary open cover of a compact set you can extract a finite subcover.

c) (Proof of Theorem 1) Let $\Omega \subset \mathbb{R}^N$ be bounded and assume that $\Omega \subset\subset \cup_{j=1}^k U_j$, with $U_j \subset \mathbb{R}^N$ open for each $j = 1, \dots, k$. Show that there exists a family $\{\eta_j\}_{j=1}^k$ with $\eta_j \in C_c^\infty(\mathbb{R}^N)$ satisfying the following properties:

- $0 \leq \eta_j \leq 1$ in $\mathbb{R}^N$, for all $j = 1, \dots, k$,
- $\operatorname{supp} \eta_j \subset U_j$ for all $j = 1, \dots, k$,
- $\sum_{j=1}^k \eta_j(x) = 1$ for all $x \in \Omega$.

Hint: use points (a) and (b). Notice that it is not restrictive to assume that the sets U_j are bounded. Also: if $\psi_1, \dots, \psi_k$ are real numbers and you define $\eta_l := \psi_l \prod_{j=1}^{l-1}(1 - \psi_j)$ for each $l = 1, \dots, k$ (with the understanding that the empty product is equal to 1), the following identity holds

$$1 - \sum_{j=1}^k \eta_j = \prod_{j=1}^k (1 - \psi_j)\,.$$