MAT.405 / WS 2019/2020
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Advanced Functional Analysis

Problem Sheet 2

Due date: October 28, 2019

I will refer to the book H.Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, 2011, Springer-Verlag New York, which you can download from the university library website.

Notation: If $\Omega \subset \mathbb{R}^N$ is an open set, we define the spaces of locally integral functions

$$L^p_{\text{loc}}(\Omega) := \{f \colon \Omega \to \mathbb{R} : \, f\chi_K \in L^p(\Omega) \ \text{ for all compact sets } \ K \subset \Omega\}$$

and of compactly supported continuous functions

$$C_c(\Omega) := \{f \in C(\Omega) : \ \exists \text{ a compact set } \ K \subset \Omega \ \text{ such that } \ f \equiv 0 \ \text{ in } \ \Omega \smallsetminus K\} \,.$$

For $k \geq 1$, we define the set of k-times continuously differentiable functions with compact support by $C^k_c(\Omega) := C^k(\Omega) \cap C_c(\Omega)$. For $f \in C^1(\Omega)$ we denote the gradient of f by $\nabla f := (\partial_{x_1} f, \dots, \partial_{x_n} f)$ where $\partial_{x_i} f := \frac{\partial f}{\partial x_1}$ is the i-th partial derivative of f. For $f \in C^k$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$ with $|\alpha| := \alpha_1 + \ldots + \alpha_N \leq k$ we denote

$$D^\alpha f := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_1^{\alpha_N}} f \,.$$

Finally the set of arbitrarily differentiable functions with compact support is denoted by $C^\infty_c(\Omega) := C^\infty(\Omega) \cap C_c(\Omega)$ where $C^\infty(\Omega) := \cap_{k=1}^\infty C^k(\Omega)$.

Convolutions: Let $N \geq 1$. Assume that $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$. We define the convolution between f and g as

$$(f \star g)(x) := \int_{\mathbb{R}^N} f(x-y) g(y) \, dy \qquad \text{for a.e. } \ x \in \mathbb{R}^N \,.$$

It should be well-known (otherwise see Theorem 4.15 Brezis) that $f \star g \in L^p(\mathbb{R}^N)$. Moreover (Proposition 4.18 Brezis)

$$\operatorname{supp}(f \star g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g} \,. \tag{1}$$

Mollifiers: let $N \geq 1$ and denote by B_r the N-dimensional ball of radius $r > 0$ centered at the origin, that is $B_r := \{x \in \mathbb{R}^N : |x| < r\}$. A sequence of *mollifiers* is any sequence of functions $\rho_n \colon \mathbb{R}^N \to \mathbb{R}$ such that, for all $n \in \mathbb{N}$,

$$\rho_n \in C^\infty_c(\mathbb{R}^N) \,, \quad \operatorname{supp} \rho_n \subset \overline{B_{1/n}} \,, \quad \rho_n \geq 0 \ \text{ on } \ \mathbb{R}^N \,, \quad \int_{\mathbb{R}^n} \rho_n(x) \, dx = 1 \,. \tag{2}$$

Assume given some function $\rho \in C^\infty_c(\mathbb{R}^N)$ such that

$$\operatorname{supp} \rho \subset \overline{B_1} \,, \quad \rho \geq 0 \ \text{ on } \ \mathbb{R}^N \,, \quad \int_{\mathbb{R}^N} \rho(x) \, dx = 1 \,.$$

It is immediate to see that the sequence $\rho_n(x) := n^N \rho(nx)$ defines a sequence of mollifiers. One standard choice for ρ is given by the map

$$\rho(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } \ |x| < 1 \,, \\ 0 & \text{if } \ |x| \geq 1 \,, \end{cases}$$

where the constant $C > 0$ is chosen so that $\int_{\mathbb{R}^N} \rho(x)\, dx = 1$.

Sobolev spaces: let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be an open set. If $f \in L^1(\Omega)$, we say that $g \in L^1(\Omega)$ is the i-th weak partial derivative of f, with $i \in \{1, \ldots, N\}$, if

$$\int_\Omega f(x)\, \partial_{x_i}\varphi(x)\, dx = -\int_\Omega g(x)\varphi(x)\, dx \quad \text{for all} \quad \varphi \in C^1_c(\Omega)\,.$$

If the i-th partial weak derivative of f exists, then it is unique (up to sets of zero Lebesgue measure). We denote such weak derivative by $g := \partial_{x_i} f$. For $1 \leq p \leq \infty$ we define the Sobolev space $W^{1,p}(\Omega)$ as

$$W^{1,p}(\Omega) := \{f \in L^p(\Omega) : \, \partial_{x_i} f \in L^p(\Omega) \text{ for all } i = 1, \ldots, N\}\,.$$

Problem 2.1 (20 pts).

a) Let $f \in C_c(\mathbb{R}^N)$ and $g \in L^1_{\text{loc}}(\mathbb{R}^N)$. Show that $f \star g$ is well defined and $f \star g \in C(\mathbb{R}^N)$.

Hint: let $x_n \to x$. Since f is compactly supported, there exists some compact set K such that $(x_n - \operatorname{supp} f) \subset K$ for all $n \in \mathbb{N}$.

b) Let $k \geq 1$, $f \in C^k_c(\mathbb{R}^N)$ and $g \in L^1_{\text{loc}}(\mathbb{R}^N)$. Show that $f \star g \in C^k(\mathbb{R}^N)$ with

$$D^\alpha(f \star g) = (D^\alpha f) \star g$$

for each multi-index α with $|\alpha| \leq k$. In particular if $f \in C^\infty_c(\mathbb{R}^N)$ then $f \star g \in C^\infty(\mathbb{R}^N)$.

Hint: by induction it is sufficient to check the statement for $k = 1$. You can directly check, using the definition of differentiability, that $\nabla(f \star g) = (\nabla f) \star g$. Notice that ∇f is uniformly continuous in $\mathbb{R}^N$, since $\operatorname{supp} \nabla f \subset \operatorname{supp} f$, and $\operatorname{supp} f$ is compact. Moreover it may be useful to recall the fundamental theorem of calculus, namely, $f(x+h) - f(x) = \int_0^1 \nabla f(x + hs) \cdot h\, ds$.

Problem 2.2 (40 pts). Let $\rho_n \colon \mathbb{R}^n \to \mathbb{R}$ be a sequence of mollifiers, so that (2) holds. Let $1 \leq p < \infty$.

a) Let $f \in C(\mathbb{R}^N)$. Show that, as $n \to \infty$, $(\rho_n \star f) \to f$ uniformly on each compact $K \subset \mathbb{R}^N$.

Hint: fix K compact. Then f is uniformly continuous on K (why?). Hence for $\varepsilon > 0$, there exists some $\delta > 0$ (depending on ε and K) such that $|f(x-y) - f(x)| < \varepsilon$ for $x \in K$, $y \in B_\delta$.

b) Let $f \in L^p(\mathbb{R}^N)$. Show that, as $n \to \infty$, $(\rho_n \star f) \to f$ strongly in $L^p(\mathbb{R}^N)$, by following the strategy below:

   i) Show that, if $f \in C_c(\mathbb{R}^N)$ then $(\rho_n \star f) \to f$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$.
   Hint: use point (a) and (1).

   ii) Show that, for $f \in L^p(\mathbb{R}^N)$, it holds that $\|\rho_n \star f\|_{L^p} \leq \|f\|_{L^p}$ for all $n \in \mathbb{N}$.
   Hint: note that $\rho_n = \rho_n^p \rho_n^{1-1/p}$ and use Hölder's inequality.

   iii) Using that $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ (Thm 4.12 Brezis), and (i)-(ii), conclude (b).

c) (Fundamental Lemma of Calculus of Variations) Let $\Omega \subset \mathbb{R}^N$ be an open set. Assume that $f \in L^1(\Omega)$ is such that

$$\int_\Omega f(x)\, \varphi(x)\, dx = 0 \quad \text{for all} \quad \varphi \in C_c(\Omega)\,.$$

Show that $f = 0$ a.e. on Ω.

Hint: first show that if $g \in L^\infty(\mathbb{R}^N)$ is such that $\operatorname{supp} g$ is compact and contained in Ω, then $\int_\Omega f\, g = 0$. This can be done by considering $g_n := \rho_n \star g$ and by using point (b), Problem 2.1 (with Ω instead of $\mathbb{R}^N$) and dominated convergence. Then apply what you just proved to some particular $L^\infty(\mathbb{R}^N)$ function in order to conclude.

The goal of the next exercise it to prove the following characterization theorem for one dimensional Sobolev functions, in the case when $\Omega = (a, b)$ is a bounded interval.

Theorem 1. Let $1 \leq p < \infty$ and $I \subset \mathbb{R}$ interval (bounded or unbounded). Let $f \in W^{1,p}(I)$. Then there exists $\tilde{f} \in C(\overline{I})$ such that $f = \tilde{f}$ a.e. in I and

$$\tilde{f}(y) - \tilde{f}(x) = \int_x^y f'(t)\,dt \quad \text{for all} \quad x, y \in \overline{I}\,.$$

The function $\tilde{f}$ is called the *continuous representative of* f.

Problem 2.3 (40 pts). Let $I = (a, b) \subset \mathbb{R}$ be a bounded interval.

a) Let $f \in L^1(I)$ be such that

$$\int_I f(x)\,\varphi'(x)\,dx = 0 \quad \text{for all} \quad \varphi \in C_c^1(I)\,.$$

Show that f is constant, i.e., $f = c$ a.e. on I for some $c \in \mathbb{R}$.

Hint: Fix $\Psi \in C_c(I)$ such that $\int_I \Psi = 1$. Then for all $w \in C_c(I)$ the map $h := w - \left(\int_I w\right)\Psi$ admits a unique continuous primitive (why?). Apply point (c) from Problem 2.2.

b) Let $g \in L^1(I)$ and define the function

$$f(x) := \int_a^x g(t)\,dt \quad \text{for} \quad x \in I\,.$$

Show that $f \in C(I)$ with $f' = g$ in the weak sense, that is,

$$\int_I f(x)\,\varphi'(x)\,dx = -\int_I g(x)\,\varphi(x)\,dx \quad \text{for all} \quad \varphi \in C_c^1(I)\,.$$

Hint: for the continuity use dominated convergence. The remaining part of the statement can be checked by employing Fubini (Theorem 4.5 Brezis).

c) Prove the statement of Theorem 1 for I bounded, with the help of (a)-(b).

Hint: study the behavior of $\bar{f}(x) := \int_a^x f'(t)\,dt$.