MAT.405 / WS 2019/2020
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Advanced Functional Analysis

Problem Sheet 1

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I will refer to the book H.Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, 2011, Springer-Verlag New York, which you can download from the university library website.

Problem 1.1 (30 pts). Assume that $\Omega \subset \mathbb{R}^n$ is an open set with $0 < |\Omega| < \infty$ and fix some p with $1 < p < \infty$. Recall that the dual of $L^p(\Omega)$ is given by $L^q(\Omega)$ with $1/p + 1/q = 1$ (Theorem 4.11 Brezis). Let $f_n \in L^p(\Omega)$ be a sequence such that

i) $f_n(x) \to f(x)$ for a.e. $x \in \Omega$ as $n \to \infty$,

ii) $\sup_n \|f_n\|_{L^p} \leq C$ for some constant $C > 0$.

The aim is to show, by means of Real Analysis tools, that $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ as $n \to \infty$. Follow the strategy below:

a) Show that $f \in L^p(\Omega)$ by using Fatou's Lemma (Lemma 4.1 Brezis).

b) Prove absolute continuity of the Lebesgue integral: namely, assume that $f \in L^1(\Omega)$. Then show that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E |f|\, dx < \varepsilon$ if $|E| < \delta$.

   Hint: by definition of Lebesgue integrability, there exists a step function $s = \sum_{j=1}^N s_j \chi_{A_j}$, with $s_j \in \mathbb{R}$, $\{A_j\}$ measurable partition of Ω, such that $0 \leq s \leq |f|$ and $\int_\Omega (|f| - s)\, dx < \varepsilon/2$.

c) Show that $f_n \rightharpoonup f$ by directly estimating the quantity

$$\int_\Omega f_n g\, dx - \int_\Omega f g\, dx\,,$$

   for some fixed $g \in L^q(\Omega)$.

   Hint: use point (b) in combination with Egoroff's Theorem (Theorem 4.29 Brezis) and Hölder's inequality (Theorem 4.6 Brezis).

Banach-Saks Property: a Banach space X is said to have the *Banach-Saks property* if for any sequence $\{x_n\} \subset X$ such that $x_n \rightharpoonup x$ weakly as $n \to \infty$, there exists a subsequence x_{n_k} of x_n such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N x_{n_k} = x$$

strongly in X. The Banach-Saks property holds when X is uniformly convex: in particular reflexive spaces, such as $X = L^p$ for $1 < p < \infty$, are uniformly convex.

Problem 1.2 (20 pts). Assume that $\Omega \subset \mathbb{R}^n$ is an open set with $0 < |\Omega| < \infty$ and fix some p with $1 < p < \infty$. Let $f_n, f \in L^p(\Omega)$ be such that (i)-(ii) from Problem 1.1. hold.

a) Show that $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ by employing the Banach-Saks property.

   Hint: you can assume as given the following facts: Banach-Alaoglu's Theorem in $L^p(\Omega)$ (Theorem 3.16 Brezis for the general statement), uniqueness of the pointwise a.e. limit, Theorem 4.9 in Brezis, and the elementary fact that if a sequence of real numbers x_n is such that $\lim_n x_n = x$, then also $\lim_N \frac{1}{N} \sum_{n=1}^N x_n = x$.

b) Show that the above claim fails for $p = 1$, for example considering the case of $L^1(0,1)$: namely, construct a sequence $f_n \in L^1(0,1)$ such that (i)-(ii) from Problem 1.1 hold, but f_n does not have any weak accumulation point.

Recall: a Banach space X is said to be reflexive if $X^{**} = j(X)$, where $j\colon X \to X^{**}$ is the canonical immersion. Also recall that the dual of L^1 is given by L^∞.

Problem 1.3 (20 pts). Show that $L^1(0,1)$ is not reflexive.

Hint: assume by contradiction that $L^1(0,1)$ is reflexive (hence Banach-Alaoglu holds) and examine the sequence $f_n(x) = n\chi_{(0,\frac{1}{n})}(x)$ to derive a contradiction. It might be useful to employ dominated convergence (Theorem 4.2 Brezis).

Recall: let H be a real Hilbert space, with scalar product denoted by $\langle \cdot, \cdot \rangle$ and induced norm $\|x\| := \sqrt{\langle x, x \rangle}$ and distance $d(x,y) := \|x - y\|$. A map $a\colon H \times H \to \mathbb{R}$ is said to be a *bilinear form* if

$$a(\lambda x + \mu y, z) = \lambda\, a(x,z) + \mu\, a(y,z)\,, \qquad a(x, \lambda y + \mu z) = \lambda\, a(x,y) + \mu\, a(x,z)$$

for all $x, y, z \in H$, $\lambda, \mu \in \mathbb{R}$. A bilinear form $a\colon H \times H \to \mathbb{R}$ is *continuous* if there exists $C \geq 0$ such that

$$|a(x,y)| \leq C\|x\|\,\|y\| \quad \text{for all} \quad x, y \in H\,,$$

and *coercive* if there exists $\alpha > 0$ such that

$$a(x,x) \geq \alpha \|x\|^2 \quad \text{for all} \quad x \in H\,.$$

Lax-Milgram's Theorem: Let H be a real Hilbert space, $a\colon H \times H \to \mathbb{R}$ a continuous and coercive bilinear form and $T\colon H \to \mathbb{R}$ a linear continuous operator. Then there exists a unique $\hat{x} \in H$ such that

$$a(\hat{x}, y) = T(y) \quad \text{for all} \quad y \in H\,. \tag{1}$$

Problem 1.4 (30 pts). Prove Lax-Milgram's Theorem by following the strategy below:

a) Show that there exists a linear continuous operator $A\colon H \to H$ such that

$$a(x,y) = \langle A(x), y \rangle \quad \text{for all} \quad x, y \in H\,.$$

Hint: for $x \in H$ fixed, the map $y \in H \mapsto a(x,y) \in \mathbb{R}$ is linear and continuous. Use Riesz Theorem (Theorem 5.5 Brezis).

b) Fix $\lambda > 0$. Show that solving (1) is equivalent to finding a fixed point $\hat{x} \in H$ for the map $S_\lambda\colon H \to H$ defined by

$$S_\lambda(x) := x - \lambda A(x) + \lambda z$$

where $z \in H$ is uniquely determined and only depends on T (we recall that a fixed point is such that $S_\lambda(\hat{x}) = \hat{x}$).

c) Show that for some $\lambda > 0$ the map S_λ is a contraction, that is, prove the existence of some $0 \leq \theta < 1$ such that

$$d(S_\lambda(x), S_\lambda(y)) \leq \theta\, d(x,y) \quad \text{for all} \quad x, y \in H\,.$$

Then conclude the proof by Theorem 5.7 Brezis.