MAT.405 / WS 2019/2020
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Advanced Functional Analysis
Exam Paper

Instructions

The exam contains 6 problems. You should choose 4 problems to solve, each graded from 0 to 25 points. You should state which problems you are going to solve. If more than 4 problems are solved, your final grade will be computed by summing the worst 4 scores. For grading, problems and sub-questions within the problems are considered independent. For example, if you only solve point (c) in Problem 1 and you have not solved (a) and (b), you will be awarded 5 points for Problem 1. The hints are there to help you, but of course you may solve the exercises in whichever way you prefer.

You may use your solutions to the problems in the Exercise Course, as well as the notes from Prof. Bredies Lectures. You have 2 hours and 30 minutes, good luck!

Problem 1. Let $N \in \mathbb{N}, N \geq 1$ and fix $1 \leq p < \infty$. For $y \in \mathbb{R}^N$, $r > 0$ denote the open ball of radius r and center at y by $B_r(y) := \{x \in \mathbb{R}^N : |x - y| < r\}$. Set $\Omega := B_1(0)$. For $\alpha > 0$ define

$$f(x) := \frac{1}{|x|^\alpha} \quad \text{for all} \quad x \in \Omega \smallsetminus \{0\}\,,$$

where $|x| := (\sum_{j=1}^N x_j^2)^{1/2}$.

a) (10 pts) Compute, in the classic sense, $\nabla f(x) := (\partial_{x_1} f(x), \dots, \partial_{x_N} f(x))$ for $x \neq 0$. Show that $f \in L^p(\Omega)$ and $\nabla f \in L^p(\Omega; \mathbb{R}^N)$ if and only if

$$\alpha < \frac{N-p}{p}\,. \tag{1}$$

b) (10 pts) Assume (1). Prove that ∇f is the weak gradient of f in Ω, in the sense that

$$\int_\Omega f\, \partial_{x_i} \phi \, dx = -\int_\Omega \partial_{x_i} f \, \phi \, dx \quad \text{for all} \quad \phi \in C_c^\infty(\Omega),\, i = 1, \dots, N\,.$$

In particular you showed that $f \in W^{1,p}(\Omega)$ if and only if (1) holds.

c) (5 pts) Assume that $\{y_k\}_{k \in \mathbb{N}}$ is a countable dense subset of Ω and that (1) holds. For a.e. $x \in \Omega$ define the sequence

$$g_n(x) := \sum_{k=1}^n \frac{1}{2^k \, |x - y_k|^\alpha}\,.$$

Show that g_n converges in $W^{1,p}(\Omega)$ as $n \to +\infty$ (you can assume that $g_n \in W^{1,p}(\Omega)$).

Hints: (a) If $f \colon \mathbb{R}^N \to \mathbb{R}$ is measurable, then $\int_{\mathbb{R}^N} f(x)\, dx = \int_0^{+\infty} \int_{\partial B_1(0)} f(rw) r^{N-1}\, d\sigma(w)\, dr$, where σ is the $(N-1)$-dimensional Hausdorff measure. Recall that $\sigma(\partial B_r(y)) = r^{N-1} \sigma_N$, where $\sigma_N := \sigma(\partial B_1(0))$ is a constant depending only on N.

(b) Let $U \subset \mathbb{R}^N$ be open, with $\partial U \in C^1$. Let ν be the outer normal to ∂U. If $f, g \in C^1(\overline{U})$ then $\int_U f\, (\partial_{x_i} g)\, dx = -\int_U (\partial_{x_i} f)\, g\, dx + \int_{\partial U} f g\, \nu_i \, d\sigma(x)$ for all $i = 1, \dots, N$. Now notice that the map given in the exercise belongs to $C^1(\overline{\Omega_\varepsilon})$ for each $\varepsilon > 0$ fixed, where $\Omega_\varepsilon := \Omega \smallsetminus B_\varepsilon(0)$.

(c) Show that g_n is a Cauchy sequence. In order to do that, try to find a uniform bound in $W^{1,p}(\Omega)$ for $|x - y_k|^{-\alpha}$.

Problem 2.

a) (10 pts) Let X be a real, reflexive Banach space. Let $J\colon X \to \mathbb{R}$ be a functional such that $J \not\equiv +\infty$ and satisfying the following properties:

   i) J is *weakly lower semicontinuous*: if $\{x_n\}$ is a sequence in X such that $x_n \rightharpoonup x$ weakly in X as $n \to +\infty$, then
   $$J(x) \leq \liminf_{n\to+\infty} J(x_n) \,.$$

   ii) J is *coercive*: if $\{x_n\}$ is a sequence in X such that $\|x_n\| \to +\infty$ as $n \to +\infty$, then
   $$\lim_{n\to+\infty} J(x_n) = +\infty \,.$$

   Show that J admits a minimum in X, that is, show that there exists a point $\hat{x} \in X$ such that
   $$J(\hat{x}) = \inf_{x\in X} J(x) \,.$$

b) (15 pts) Let $d \in \mathbb{N}, d \geq 1$ and $\Omega \subset \mathbb{R}^d$ be open and bounded. Consider the Hilbert space $H_0^1(\Omega)$ equipped with the norm $\|u\|_{H_0^1(\Omega)} := \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega;\mathbb{R}^d)}$. Fix $f \in L^2(\Omega)$ and define the functional $J\colon H_0^1(\Omega) \to \mathbb{R}$ by
   $$J(u) := \int_\Omega |\nabla u|^2 \, dx - \int_\Omega f u \, dx \,.$$

   By verifying the assumptions in point (a), prove that J has a minimizer $\hat{u}$ in $H_0^1(\Omega)$.

Hints: (a) Take a minimizing sequence, i.e., $\{x_n\} \subset X$ such that $\lim_n J(x_n) = \inf_{x\in X} J(x)$. What can you say about its accumulation points?
(b) Remember that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ if and only if $u_n \to u$ strongly in $L^2(\Omega)$ and $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^d)$. Also recall Poincaré inequality: there exists a constant $C > 0$ (depending only on Ω) such that $\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega;\mathbb{R}^d)}$ for all $u \in H_0^1(\Omega)$.

Problem 3. Let $X := C[0,1]$. On X we consider two topologies τ_d and τ_p. The topology τ_d is induced by the metric $d\colon X \times X \to [0,\infty)$ defined by
$$d(f,g) := \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} \, dx \quad \text{for} \quad f,g \in X \,.$$

The topology τ_p is induced by the family of seminorms $\{p_x\colon x \in [0,1]\}$ where $p_x(f) := |f(x)|$ for $f \in X, x \in [0,1]$, making X into a LCS. Consider the identity map
$$I\colon (X,\tau_p) \to (X,\tau_d) \ \text{ where } \ I(f) := f \ \text{ for } \ f \in X \,.$$

The goal of this exercise is to show that (X,τ_p) is not metrizable, by proving that I is sequentially continuous but not continuous.

a) (10 pts) Show that I is sequentially continuous.

b) (15 pts) Prove that I is not continuous.

Hints: (a) Notice that τ_p is the topology of pointwise convergence, in the sense that if $f_n \overset{\tau_p}{\to} f$, then $f_n \to f$ pointwise in $[0,1]$.
(b) Show that I is not continuous at $f = 0$, arguing by contradiction. Start by recalling the definition of open sets in a locally convex space. If V is open in (X,τ_d), then $I^{-1}(V) = V$ is open in (X,τ_p).

Problem 4. Consider $C[0,1]$ equipped with the supremum norm $\|f\|_\infty := \sup_x |f(x)|$. We denote constant functions by $f \equiv c$. The goal of this exercise is to show that $(C[0,1], \|\cdot\|_\infty)$ is not the dual of any Banach space X.

a) (20 pts) Consider the convex unit ball of $C[0,1]$

$$K := \{f \in C[0,1] : \; \|f\|_\infty \leq 1\} .$$

Show that

$$\mathrm{ext}(K) = \{f \in C[0,1] : \; f \equiv 1 \;\; \text{or} \;\; f \equiv -1\} ,$$

that is, the only extremal points of K are given by the constant functions with values ± 1.

b) (5 pts) By using point (a) and Krein-Milman's Theorem, show that $C[0,1]$ is not the dual of any Banach space X.

Hints: (a) Assume that $f \equiv \pm 1$. In order to see that $f \in \mathrm{ext}(K)$, it is useful to notice that if $a, b \in \mathbb{R}$ and $a < b$, then $\mathrm{co}\{a, b\} = [a, b]$.
Conversely, let $f \in \mathrm{ext}(K)$. Assume by contradiction that $|f|$ is not constantly equal to 1. Notice that for all $g \in C[0,1]$ you always have the decomposition $f = (f+g)/2 + (f-g)/2$. Try to construct a function g such that the previous decomposition is non-trivial.
(b) You can use that $\mathrm{co}(\mathrm{ext}(K)) = \{f \in C[0,1] : \; f \equiv c \;\text{ for some }\; |c| \leq 1\}$ and that this set is isometric to the unit ball of $\mathbb{R}$. Therefore all topologies are equivalent on $\mathrm{co}(\mathrm{ext}(K))$.

Problem 5. Consider the problem of finding $\Lambda \in \mathcal{D}(\mathbb{R})^*$ such that the equation

$$x\Lambda' + \Lambda = 0 \tag{2}$$

is satisfied in the sense of $\mathcal{D}(\mathbb{R})^*$. The goal of this exercise is to show that all the distributional solutions to (2) are of the form

$$\Lambda = a \,\mathrm{PV}\,\frac{1}{x} + b\,\delta \,, \tag{3}$$

for some $a, b \in \mathbb{R}$. Recall that for $\phi \in \mathcal{D}(\mathbb{R})$ we define

$$\delta\phi := \phi(0)\,, \qquad \left(\mathrm{PV}\,\frac{1}{x}\right)(\phi) := \lim_{\varepsilon \to 0} \int_{\{|x| \geq \varepsilon\}} \frac{\phi(x)}{x}\,dx \,.$$

a) (10 pts) Let $\Lambda \in \mathcal{D}(\mathbb{R})^*$. Prove the constancy theorem, that is, show that they are equivalent:

   i) $\Lambda' = 0$ in $\mathcal{D}(\mathbb{R})^*$,

   ii) There exists $c \in \mathbb{R}$ such that $\Lambda = c$, that is, $\Lambda\phi = c \int_{\mathbb{R}} \phi(x)\,dx$ for all $\phi \in \mathcal{D}(\mathbb{R})$.

b) (8 pts) Let $\Lambda \in \mathcal{D}(\mathbb{R})^*$ be of the form (3) for some $a, b \in \mathbb{R}$. Show that Λ solves (2).

c) (7 pts) Assume that $\Lambda \in \mathcal{D}(\mathbb{R})^*$ is a solution to (2). Show that Λ is of the form (3) for some $a, b \in \mathbb{R}$.

Hints: (a) You can use the following fact: If $\phi \in \mathcal{D}(\mathbb{R})$, then $\int_{\mathbb{R}} \phi\,dx = 0$ if and only if there exists $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi' = \phi$. Also notice that, if $\Lambda = c$, then necessarily $c = \Lambda\phi_0$ for every $\phi_0 \in \mathcal{D}(\mathbb{R})$ such that $\int_{\mathbb{R}} \phi_0\,dx = 1$.
(b) Notice that $(x\Lambda)' = x\Lambda' + \Lambda$ by the Leibniz rule.
(c) You can use the following: If $\Lambda \in \mathcal{D}(\mathbb{R})^*$ is such that $x\Lambda = 0$, then $\Lambda = c\delta$ for some $c \in \mathbb{R}$.

Problem 6. Set $X := C[0,1]$ equipped with the supremum norm. Fix $\varphi \in X$ and define the linear operator $M\colon X \to X$ such that

$$(Mf)(x) := \varphi(x)f(x) \quad \text{for all} \quad x \in [0,1],\ f \in X\,.$$

a) (5 pts) Show that $M \in \mathcal{L}(X)$ and compute $\|M\|$.

b) (10 pts) Show that M is compact if and only if $\varphi \equiv 0$.

c) (10 pts) Let $R(\varphi) := \{\varphi(x)\,,\ x \in [0,1]\}$ be the range of φ. For $\lambda \in \mathbb{R}$, denote the λ-level set of φ by $\varphi^{-1}(\lambda) := \{x \in [0,1] :\ \varphi(x) = \lambda\}$. Prove that the resolvent of M is characterized by

$$\rho(M) = \mathbb{R} \setminus R(\varphi)\,.$$

Moreover show that the eigenvalues of M are given by

$$\sigma_p(M) = \{\lambda \in \mathbb{R} :\ \exists\ 0 \le a < b \le 1 \ \text{ such that } \ (a,b) \subset S_\lambda\}\,.$$

Hints: (b) You can try to prove this directly, by using Ascoli-Arzelà: If M is compact, then the set $\{\varphi f :\ f \in C[0,1],\ \|f\| \le 1\}$ is relatively compact and hence equicontinuous. From this fact, by constructing a suitable sequence $f_n \in C[0,1]$ with $\|f_n\| \le 1$, you can conclude that $\varphi \equiv 0$.