Optimal transport regularization of dynamic inverse problems

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(joint work with Kristian Bredies)

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Motivation: Motion-Aware Tomographic Reconstruction

Motion on sub-acquisition time scales $\rightsquigarrow$ **artefacts** in reconstructed images

- Imaging of lung or heart (motion cannot be suppressed)
- High-resolution imaging (sub-millimeter motion poses problems)

Workarounds: use of anaesthetics, breath-holding strategies, gating

Drawbacks: assumes periodicity (arrhythmias?). Still limited to low-resolution

Reference image

No regularizer

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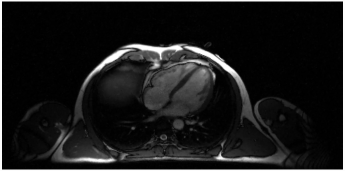

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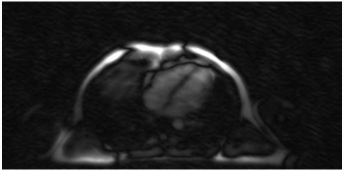


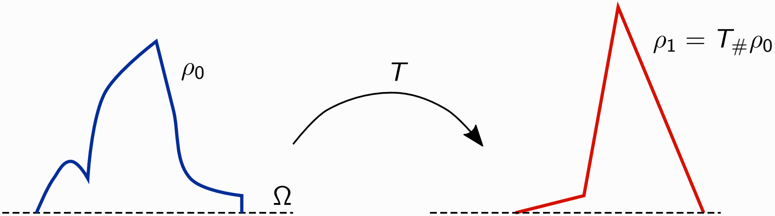
No regularizer

Proposed model: optimal transport regularization for dynamic reconstruction

K. Bredies, S. Fanzon - An optimal transport approach for solving dynamic inverse problems in spaces of measures. Preprint 2019

Optimal Transport - Static Formulation

$\Omega \subset \mathbb{R}^d$ bounded domain, $\rho_0, \rho_1 \in \mathcal{P}(\Omega)$, $T \colon \Omega \to \Omega$ measurable displacement



Goal: move ρ_0 to ρ_1 in the cheapest way, with cost of moving mass from x to y

$$c(x, y) := |x - y|^2$$

Optimal Transport: a transport plan T solving

$$\min \left\{ \int_\Omega |T(x) - x|^2 \, d\rho_0(x) : \; T \colon \Omega \to \Omega, \; T_{\#}\rho_0 = \rho_1 \right\}$$

Optimal Transport - Dynamic Formulation

Idea: introduce a time variable $t \in [0,1]$ and consider evolution of ρ_t

- time dependent probability measures

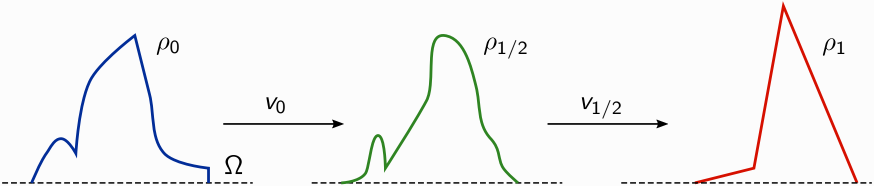
$$t \mapsto \rho_t \in \mathcal{P}(\Omega) \quad \text{for} \quad t \in [0,1]$$

- velocity field advecting ρ_t

$$v_t(x) \colon [0,1] \times \Omega \to \mathbb{R}^d$$

- (ρ_t, v_t) solves the continuity equation with initial conditions

$$\begin{cases} \partial_t \rho_t + \operatorname{div}(\rho_t v_t) = 0 \\ \text{Initial data } \rho_0, \text{ final data } \rho_1 \end{cases} \tag{CE-IC}$$



Connection and Advantages

Theorem (Benamou-Brenier '00)

$$\min_{\substack{(\rho_t, v_t) \\ \textit{solving (CE-IC)}}} \int_0^1 \int_\Omega |v_t(x)|^2 \rho_t(x) dx\, dt = \min_{\substack{T:\, \Omega \to \Omega \\ T_\# \rho_0 = \rho_1}} \int_\Omega |T(x) - x|^2 \rho_0(x)\, dx$$

Advantages of Dynamic Formulation:

1. By introducing the momentum $m_t := \rho_t v_t$ we have

$$\int_0^1 \int_\Omega |v_t(x)|^2 \rho_t(x)\, dx\, dt = \int_0^1 \int_\Omega \frac{|m_t(x)|^2}{\rho_t(x)}\, dx\, dt$$

   which is **convex** in (ρ_t, m_t). The continuity equation becomes **linear**

$$\partial_t \rho_t + \operatorname{div} m_t = 0$$

2. we know the full trajectory ρ_t and can recover the velocity field v_t from m_t

Dynamic inverse problem

$\Omega \subset \mathbb{R}^d$ bounded open domain, $d \geq 1$

For $t \in [0,1]$ assume given

- H_t Hilbert spaces (measurement spaces - non isomorphic)
- $K_t^* \colon \mathcal{M}(\overline{\Omega}) \to H_t$ linear continuous operators (forward-operators)

Time dependence allows for spatial undersampling - e.g. line or point sampling

Problem

Given some data $\{f_t\}_{t\in[0,1]}$ with $f_t \in H_t$, find a curve of measures

$$t \mapsto \rho_t \in \mathcal{M}(\overline{\Omega})$$

such that they solve the dynamic inverse problem

$$K_t^* \rho_t = f_t \qquad \text{for a.e.} \quad t \in [0,1]\,. \tag{P}$$

Optimal transport regularization

Consider a triple $(\rho_t,\, v_t,\, g_t)$ with

- $v_t \colon (0,1) \times \overline{\Omega} \to \mathbb{R}^d$ velocity field
- $g_t \colon (0,1) \times \overline{\Omega} \to \mathbb{R}$ growth rate

We propose to regularize (P) via minimization in $(\rho_t,\, v_t,\, g_t)$ of

$$\underbrace{\frac{1}{2}\int_0^1 \|K_t^* \rho_t - f_t\|_{H_t}^2 \, dt}_{\text{Fidelity Term}} + \frac{\alpha}{2} \underbrace{\int_0^1 \int_{\overline{\Omega}} |v_t(x)|^2 + |g_t(x)|^2 \, d\rho_t(x) dt}_{\text{Optimal Transport Regularizer}} + \beta \underbrace{\int_0^1 \|\rho_t\| \, dt}_{\text{TV Regularizer}}$$

$$\text{s.t.} \quad \partial_t \rho + \operatorname{div}(\rho v_t) = \rho g_t \qquad \text{(Continuity Equation)}$$

- v_t keeps track of motion
- g_t allows the presence of a contrast agent
- continuity equation enforces "regular" motion

Formal definition of the OT Energy

Set $X := (0,1) \times \overline{\Omega}$ and consider triples $(\rho, m, \mu) \in \mathcal{M}(X)^{d+2}$

Define the convex, 1-homogeneous functional

$$B(\rho, m, \mu) := \int_X \Psi\left(\frac{d\rho}{d\lambda}, \frac{dm}{d\lambda}, \frac{d\mu}{d\lambda}\right) d\lambda$$

where $\lambda \in \mathcal{M}^+(X)$ is such that $\rho, m, \mu \ll \lambda$ and

$$\Psi(t,x,y) := \frac{x^2 + |y|^2}{2t} \quad \text{if} \quad t > 0, \ \ \Psi = +\infty \text{ else}$$

Proposition (Fanzon, Bredies '19)

B is weak lower-semicontinuous. If $B(\rho, m, \mu) < +\infty$ and $\partial_t \rho + \operatorname{div} m = \mu$ then*

- *$\rho = dt \otimes \rho_t$ for a weak*-continuous curve $t \mapsto \rho_t \in \mathcal{M}^+(\overline{\Omega})$*
- *$m = \rho v_t$ for some velocity field $v_t \colon (0,1) \times \overline{\Omega} \to \mathbb{R}^d$*
- *$\mu = \rho g_t$ for some growth rate $g_t \colon (0,1) \times \overline{\Omega} \to \mathbb{R}$*

$$B(\rho, m, \mu) = \int_0^1 \int_{\overline{\Omega}} |v_t(x)|^2 + |g_t(x)|^2 \, d\rho_t(x) \, dt$$

Formal definition of the regularizer

Assumptions on H_t and K_t^*:

(H) the spaces H_t vary in a "measurable" way $\rightsquigarrow$ possible to define Hilbert space

$$L^2_H := \left\{ f\colon [0,1] \to \cup_t H_t : \; f_t \in H_t \text{ “strongly measurable”}, \int_0^1 \|f_t\|^2_{H_t} \, dt < \infty \right\}$$

(K) the operators $K_t^* \colon \mathcal{M}(\overline{\Omega}) \to H_t$ are weak*-to-weak continuous and

- $\sup_t \|K_t^*\| \leq C$
- for $\rho \in \mathcal{M}(\overline{\Omega})$ the map $t \mapsto K_t^* \rho$ is strongly measurable

Proposition: if $t \mapsto \rho_t \in \mathcal{M}(\overline{\Omega})$ is weak*-continuous then $(t \mapsto K_t^* \rho_t) \in L^2_H$

Definition (Regularization)

Let $f \in L^2_H$. For $(\rho, m, \mu) \in \mathcal{M}(X)^{d+2}$ such that $\partial_t \rho + \operatorname{div} m = \mu$ define

$$J_{\alpha,\beta}(\rho, m, \mu) := \frac{1}{2} \int_0^1 \|K_t^* \rho_t - f_t\|^2_{H_t} \, dt + \alpha \, B(\rho, m, \mu) + \beta \, \|\rho\|_{\mathcal{M}(X)} \,.$$

Existence & Regularity

Theorem (Existence)

Assume (H)-(K) and let $f \in L^2_H$. Then

$$\min_{(\rho,m,\mu)\in\mathcal{M}} J_{\alpha,\beta}(\rho,m,\mu) \qquad \text{(MIN)}$$

admits a solution. If K_t^* is injective for a.e. t, then the solution is unique.

Theorem (Regularity)

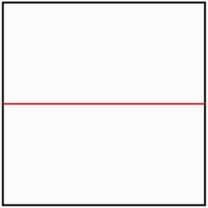
Assume (H)-(K). Let f^n be noisy data such that $f^n \to f^\dagger$ in L^2_H.
If (ρ^n, m^n, μ^n) is a minimizer of (MIN) with par. $\alpha_n, \beta_n \to 0$ and data f^n, then

$$(\rho^n, m^n, \mu^n) \overset{*}{\rightharpoonup} (\rho^\dagger, m^\dagger, \mu^\dagger)\,, \quad K_t^*\rho_t^\dagger = f_t^\dagger \quad \textit{for all} \quad t \in [0,1]$$

$$(\rho^\dagger, m^\dagger, \mu^\dagger) \in \arg\min\, \alpha^* B(\rho,m,\mu) + \beta^* \|\rho\|_{\mathcal{M}(X)}\,,\ \exists\, \alpha^*, \beta^* \geq 1$$

Variational reconstruction for undersampled MRI

- $\Omega = [0,1]^2$ image frame, $t \mapsto \rho_t \in \mathcal{M}(\overline{\Omega})$ proton density
- $H_t := L^2_{\sigma_t}(\mathbb{R}^2; \mathbb{C}^N)$ with $\sigma_t \in \mathcal{M}^+(\mathbb{R}^2)$ sampling measures



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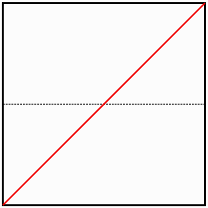

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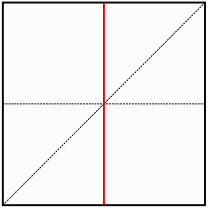

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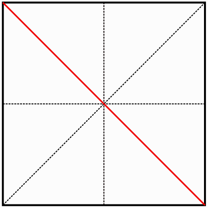

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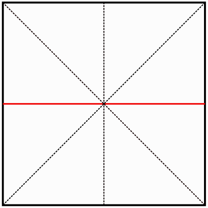

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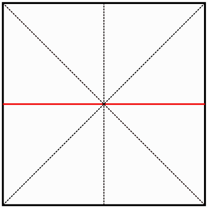

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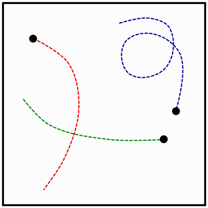

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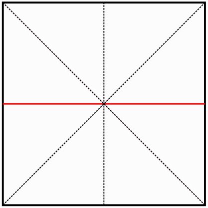

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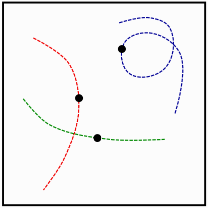

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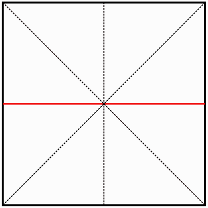

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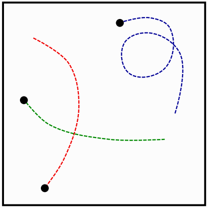

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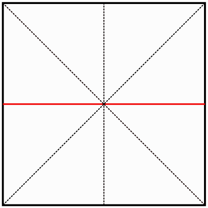

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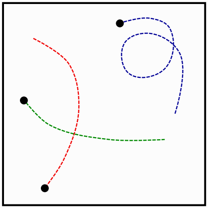

$\sigma_t = \mathcal{H}^0 \llcorner P_t$

- $K_t^* \colon \mathcal{M}(\overline{\Omega}) \to H_t$ masked Fourier transform

$$K_t^* \rho := (\mathfrak{F}(c_1 \rho), \dots, \mathfrak{F}(c_N \rho))$$

with $c_j \in C_0(\mathbb{R}^2; \mathbb{C})$ coil sensitivities (accounting for phase inhomogeneities)

Variational reconstruction for undersampled MRI

Assumptions on σ_t:

- (M1): $\sup_t \|\sigma_t\| \leq C$
- (M2): for each $\varphi \in C_0(\mathbb{R}^2;\mathbb{C})$ the map $t \mapsto \int_{\mathbb{R}^2} \varphi(x)\, d\sigma_t(x)$ is measurable

Theorem (Fanzon, Bredies '19)

Assume (M1)-(M2). Let $\alpha, \beta, \delta > 0$, $f \in L^2([0,1];H)$ *and* $c \in C_0(\mathbb{R}^2;\mathbb{C}^N)$. *Then*

$$\min_{\substack{(\rho,m,\mu) \\ \partial_t \rho + \operatorname{div} m = \mu}} \frac{1}{2} \sum_{j=1}^{N} \int_0^1 \|\mathfrak{F}(c_j \rho_t) - f_t\|^2_{L^2_{\sigma_t}} \, dt + \alpha B_\delta(\rho, m, \mu) + \beta \|\rho\|$$

admits a solution (ρ, m, μ) *with*

- $\rho = dt \otimes \rho_t$ *with* $t \mapsto \rho_t$ *weak* continuous*
- $m = \rho v$ *for some velocity* $v\colon (0,1) \times \overline{\Omega} \to \mathbb{R}^2$
- $\mu = \rho g$ *for some growth rate* $g\colon (0,1) \times \overline{\Omega} \to \mathbb{R}^2$

Extremal Points

Consider the regularizer for the homogenous case (no source): $(\rho, m) \in \mathcal{M}(X)^{d+1}$

$$R_{\alpha,\beta}(\rho, m) := \alpha B(\rho, m) + \beta \|\rho\|_{\mathcal{M}(X)} \ \text{ s.t. } \ \partial_t \rho + \operatorname{div} m = 0$$

Recall: if $m = v\rho$ and $\rho = dt \otimes \rho_t$

$$R_{\alpha,\beta}(\rho, m) := \frac{\alpha}{2} \int_0^1 |v(t,x)|^2 \, d\rho_t(x) \, dt + \beta \int_0^1 \|\rho_t\|_{\mathcal{M}(\overline{\Omega})} \, dt$$

Theorem (Fanzon, Bredies, Carioni, Romero '19)

Let $C := \{(\rho, m) : \ R_{\alpha,\beta}(\rho, m) \leq 1\}$. *Then*

$$\mathrm{Ext}(C) = \{(0,0)\} \cup \mathcal{C}$$

where

$$\mathcal{C} := \left\{ (\rho_\gamma, m_\gamma) : \ \gamma \in \mathrm{AC}^2([0,1]; \overline{\Omega}) \right\}$$

$$\rho_\gamma := a_\gamma \, dt \otimes \delta_{\gamma(t)} \, , \ m_\gamma := \dot{\gamma} \, \rho_\gamma \, , \ a_\gamma^{-1} := \frac{\alpha}{2} \int_0^1 |\dot{\gamma}(t)|^2 \, dt + \beta$$

L. Ambrosio. Inventiones mathematicae, 158(2) '04

Discrete time sampling and finite-dimensional data

Fix $N \geq 1$ times $0 < t_1 < t_2 < \cdots < t_N < 1$ and let

- H_i finite dimensional Hilbert space, $\mathcal{H} := \bigtimes_{i=1}^N H_i$
- $K_i^* \colon \mathcal{M}(\overline{\Omega}) \to H_i$ linear and weak*-continuous

Inverse problem: for $(f_1, \ldots, f_N) \in \mathcal{H}$ find a curve $t \mapsto \rho_t \in \mathcal{M}(\overline{\Omega})$ such that

$$K_i^* \rho_{t_i} = f_i \quad \text{for} \quad i = 1, \ldots, N$$

Regularization: we regularize with

$$J_{\alpha,\beta}(\rho, m) := \frac{1}{2} \sum_{i=1}^{N} \|K_i \rho_{t_i} - f_i\|_{H_i}^2 + \alpha B(\rho, m) + \beta \|\rho\|_{\mathcal{M}(X)}$$

Sparse minimizers

Theorem (Fanzon, Bredies, Carioni, Romero '19)

The minimization problem

$$\min_{(\rho,m)\in\mathcal{M}} \frac{1}{2}\sum_{i=1}^{N} \|K_i\rho_{t_i} - f_i\|_{H_i}^2 + \alpha B(\rho, m) + \beta \|\rho\|_{\mathcal{M}(X)}$$

admits a sparse minimizer of the form

$$(\rho^*, m^*) = \sum_{i=1}^{p} c_i \,(\rho_{\gamma_i}, m_\gamma)$$

where $c_i > 0$, $\gamma_i \in \mathrm{AC}^2([0,1];\overline{\Omega})$ *and* $p \leq \dim \mathcal{H}$.

K. Bredies, M. Carioni '18

C. Boyer, A. Chambolle, Y. De Castro, V. Duval, F. De Gournay, P. Weiss '18

Conclusions and Perspectives

Conclusions:

- Introduced rigorous framework for optimal transport regularization of time dependent inverse problems
- Application to variational reconstruction for undersampled MRI
- Characterization of the extremal points of the regularizer
- Existence of sparse minimizers for discrete time sampling and finite dimensional data spaces

Perspectives:

- Numerical algorithms for dynamic spike reconstruction (in progress...)
  (based on knowledge of extremal points and conditional gradient methods)
- Extremal points for the non-homogeneous case and numerics (in progress...)

Thank You!