# Geometric Patterns and Microstructures in the study of Material Defects and Composites

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# Presentation Plan

## 1 Geometric Patterns of Dislocations

- Dislocations
- Semi-coherent interfaces (Chapter 3)

  F., Palombaro, Ponsiglione. *A Variational Model for Dislocations at Semi-coherent Interfaces.* Journal of Nonlinear Science (2017)
- Linearised polycrystals (Chapter 4)

  F., Palombaro, Ponsiglione. *Linearized Polycrystals from a 2D System of Edge Dislocations.* Preprint (2017)

## 2 Microstructures in Composites

- Critical lower integrability (Chapter 5)

  F., Palombaro. *Optimal lower exponent for the higher gradient integrability of solutions to two-phase elliptic equations in two dimensions.* Calculus of Variations and Partial Differential Equations (2017)
- Convex integration
- Proof of the main theorem

# Presentation Plan

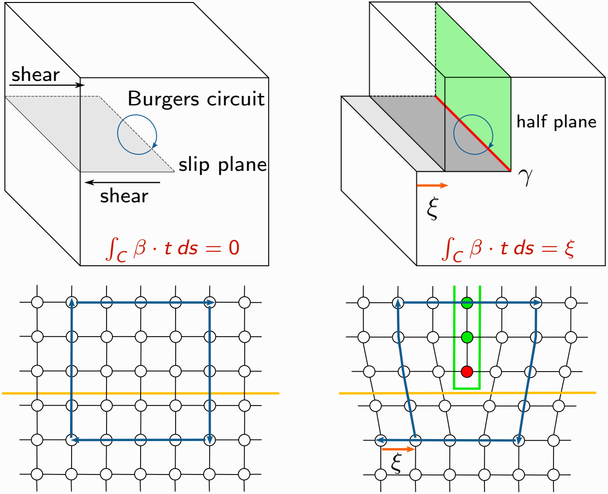
1. **Geometric Patterns of Dislocations**
   - Dislocations
   - Semi-coherent interfaces
   - Linearised polycrystals

2. **Microgeometries in Composites**
   - Critical lower integrability
   - Convex integration
   - Proof of our main result

# Edge dislocations

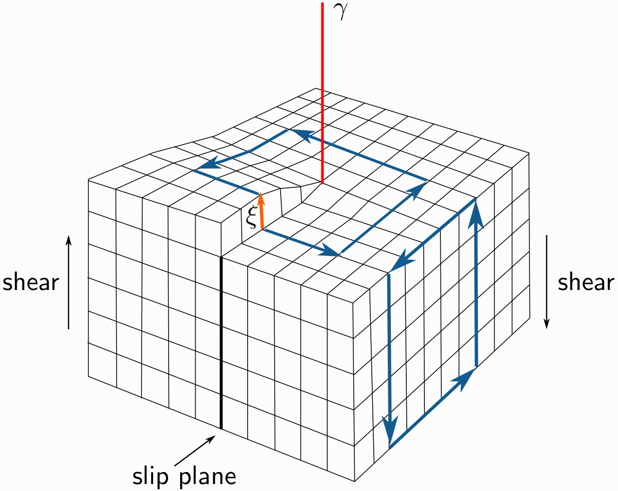
**Dislocations:** topological defects in the otherwise periodic structure of a crystal.

**Edge dislocation:** pair $(\gamma, \xi)$ of dislocation line and Burgers vector, with $\xi \perp \gamma$.



# Screw dislocations

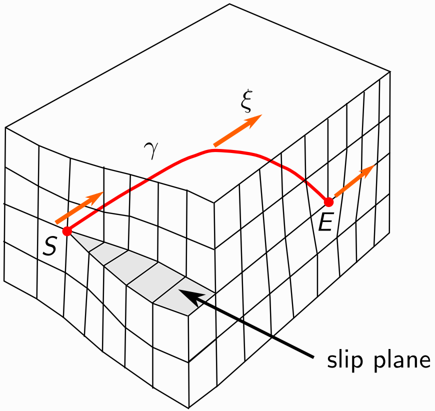
**Screw dislocation:** pair $(\gamma, \xi)$ of dislocation line and Burgers vector, with $\xi /\!/ \gamma$.

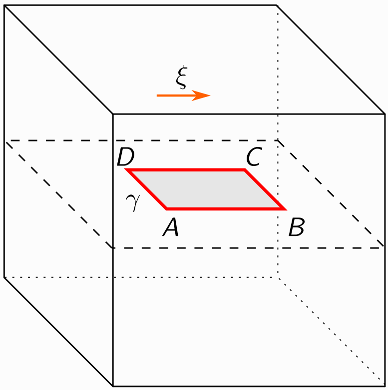


# Mixed type dislocations

**Mixed dislocations:** Burgers vector $\xi$ is constant and $\gamma$ is curved.

**Dislocation type:** given by the angle between $\xi$ and $\dot{\gamma}$.



# Nonlinear Elasticity

**Reference configuration:** $\Omega \subset \mathbb{R}^3$ open bounded

**Deformations:** regular maps $v \colon \Omega \to \mathbb{R}^3$

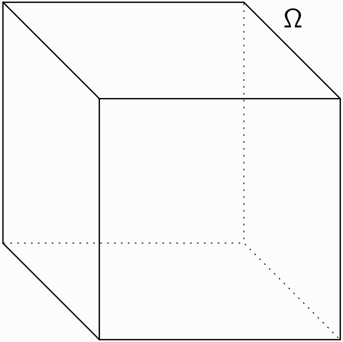
**Deformation strain:** $\beta := \nabla v \colon \Omega \to \mathbb{M}^{3\times 3}$

**Energy:** associated to a deformation strain $\beta$

$$E(\beta) := \int_\Omega W(\beta)\, dx\,.$$

**Energy Density:** $W \colon \mathbb{M}^{3\times 3} \to [0,\infty)$ s.t.

- $W$ is continuous
- $W(F) = W(RF)$, $\forall R \in SO(3), F \in \mathbb{M}^{3\times 3}$ (frame indifferent),
- $W(F) \sim \mathrm{dist}(F, SO(3))^2 \implies W(I) = 0.$



# Semi-discrete model for dislocations

**Dislocation lines:** Lipschitz curves $\gamma \subset \Omega$ such that $\Omega \setminus \gamma$ is not simply connected

**Burgers vector:** $\xi \in \mathcal{S}$ set of slip directions

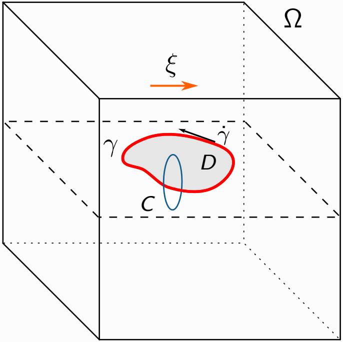
**Strain generating $(\gamma, \xi)$:** map $\beta \colon \Omega \to \mathbb{M}^{3\times 3}$ s.t.

$$\operatorname{Curl} \beta = -\xi \otimes \dot{\gamma}\, \mathcal{H}^1 \llcorner \gamma \iff \int_C \beta \cdot t \, d\mathcal{H}^1 = \xi \,.$$

**Geometric interpretation:** if $D$ encloses $\gamma$, there exists a deformation $v \in SBV(\Omega; \mathbb{R}^3)$ s.t.

$$Dv = \nabla v \, dx + \xi \otimes n \, \mathcal{H}^2 \llcorner D \,, \quad \beta = \nabla v \,.$$

$v$ has constant jump $\xi$ across the slip region $D$.



# Strains are not $L^2$

Let $\beta$ generate $(\gamma, \xi)$. Consider $\varepsilon > 0$ and

$$I_\varepsilon(\gamma) := \{x \in \mathbb{R}^3 : \ \mathrm{dist}(x,\gamma) < \varepsilon\}\,.$$

Then we have

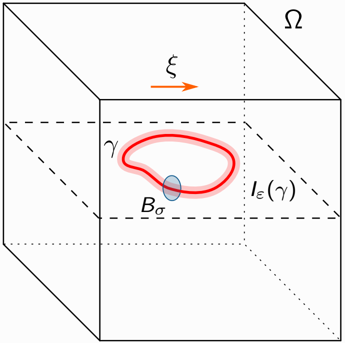
$$|\beta(x)| \sim \frac{1}{\mathrm{dist}(x,\gamma)} \ \text{ in } \ I_\varepsilon(\gamma) \implies \beta \notin L^2(I_\varepsilon(\gamma))$$

**Proof:** let $\sigma > \varepsilon$ and $L := \mathrm{length}(\gamma)$

$$\int_{I_\sigma \setminus I_\varepsilon} |\beta|^2 = L \int_\varepsilon^\sigma \int_{\partial B_\rho(\gamma(s))} |\beta|^2 \, d\mathcal{H}^1 \, d\rho$$

$$(\text{Jensen}) \geq L \int_\varepsilon^\sigma \frac{1}{2\pi\rho} \left| \int_{\partial B_\rho(\gamma(s))} \beta \cdot t \, d\mathcal{H}^1 \right|^2 d\rho$$

$$= L \frac{|\xi|^2}{2\pi} \log \frac{\sigma}{\varepsilon} \to \infty \ \text{ as } \ \varepsilon \to 0$$



# Regularise the problem

**Energy Truncation.** Fix $p \in (1,2)$ and assume

$$W(F) \sim \operatorname{dist}(F, SO(3))^2 \wedge (|F|^p + 1)\,.$$

Strains are maps $\beta \in L^2(\Omega; \mathbb{M}^{3\times 3})$ such that

$$\operatorname{Curl} \beta = -\xi \otimes \dot\gamma\, \mathcal{H}^1 \llcorner \gamma\,.$$

**Core Radius Approach.** Assume

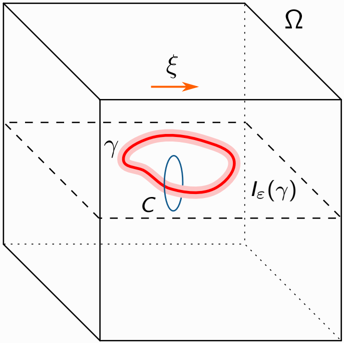
$$W(F) \sim \operatorname{dist}(F, SO(3))^2\,.$$

Let $\varepsilon > 0$ ($\propto$ atomic distance) and consider

$$\Omega_\varepsilon(\gamma) := \Omega \setminus I_\varepsilon(\gamma)\,.$$

Strains are maps $\beta \in L^2(\Omega_\varepsilon(\gamma); \mathbb{M}^{3\times 3})$ such that

$$\operatorname{Curl} \beta \llcorner \Omega_\varepsilon(\gamma) = 0\,, \quad \int_C \beta \cdot t \, d\mathcal{H}^1 = \xi\,.$$



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1. **Geometric Patterns of Dislocations**
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   - Semi-coherent interfaces
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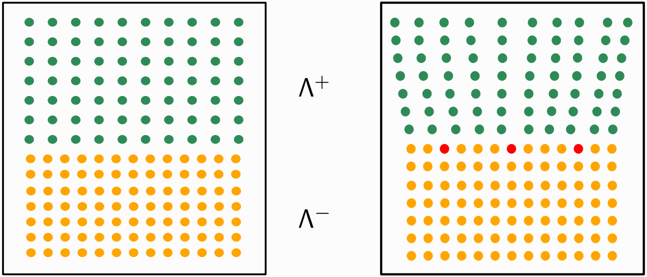
# Semi-coherent interfaces

Two different crystalline materials joined at a flat interface:

- **Underlayer:** cubic lattice $\Lambda^-$, spacing $b > 0$ (equilibrium $I$),
- **Overlayer:** lattice $\Lambda^+ = \alpha\Lambda^-$, with $\alpha > 1$ (not in equilibrium).

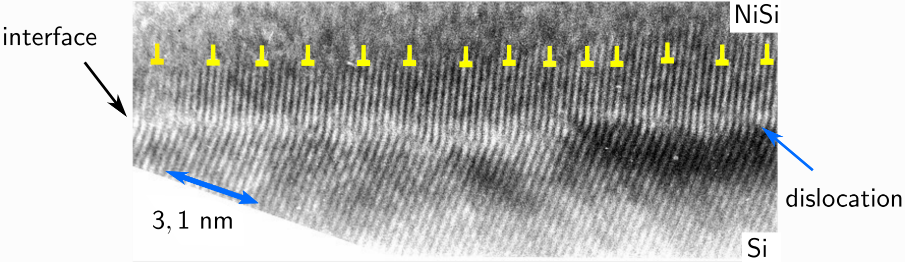
**Semi-coherent interface:** small dilation $\boldsymbol{\alpha \approx 1}$.

**Equilibrium:** $\Lambda^+$ has lower density than $\Lambda^-$ $\implies$ **edge dislocations** at interface.



# Network of dislocations

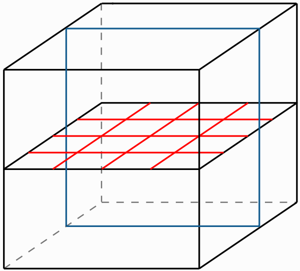
**Experimentally observed phenomena:**

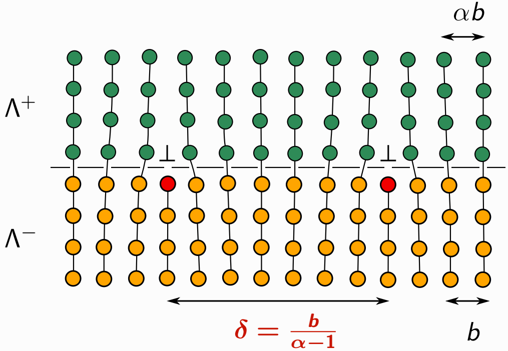


# Network of dislocations

**Experimentally observed phenomena:**

- two non-parallel sets of **edge dislocations** with spacing $\delta = \frac{b}{\alpha - 1}$,
- far field stress is completely relieved.






D.A. Porter, K.E. Easterling. *Phase transformations in metals and alloys.* CRC Press (2009)

G. Gottstein. *Physical foundations of materials science.* Springer (2013)

# Goal of the Paper

$R$ is the size of the interface.

**Goal:** define a **continuum model** such that

- $\exists$ critical size $R^*$ such that nucleation of dislocations is energetically more favorable for $\boldsymbol{R > R^*}$,
- as $R \to \infty$ the far field stress is relieved,
- the dislocation spacing tends to $\boldsymbol{\delta = \dfrac{b}{\alpha - 1}}$.

**Plan:**

- analysis of a **semi-discrete model** where dislocations are line defects,
- derive the simplified (dislocation density) **continuum model**.

F., Palombaro, Ponsiglione. *A Variational Model for Dislocations at Semi-coherent Interfaces.* Journal of Nonlinear Science (2017)

# Semi-discrete line defect model

**Reference configuration:** $\Omega_r := \Omega_r^- \cup S_r \cup \Omega_r^+,\ r > 0,$

- $\Omega_r^+$ overlayer (equilibrium $\alpha I$),
- $\Omega_r^-$ underlayer (in equilibrium and rigid).

**Energy density:** $W\colon \mathbb{M}^{3\times 3} \to [0,\infty)$ continuous, s.t.

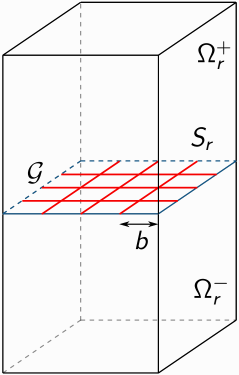
- $W(F) = W(RF)$, $\forall R \in SO(3)$ (frame indifference),
- $W(F) \sim \operatorname{dist}(F, \alpha SO(3))^2 \wedge (|F|^p + 1)$ for $1 < p < 2$.

**Admissible dislocations:** compatible with cubic lattice. $(\Gamma, B) \in \mathcal{AD}$ if $\Gamma = \{\gamma_i\}$, $B = \{\xi_i\}$ with

- dislocation line $\gamma_i \subset \mathcal{G}$ relatively closed,
- Burgers vector $\xi_i \in b(\mathbb{Z} \oplus \mathbb{Z})$.

**Admissible strains:** for a dislocation $(\Gamma, B)$ are the maps $\beta \in AS(\Gamma, B)$, such that $\beta \in L^p(\Omega_r; \mathbb{M}^{3\times 3})$ and

$$\beta = I \ \text{ in } \ \Omega_r^-\,, \qquad \operatorname{Curl}\beta = -\xi \otimes \dot\gamma\, \mathcal{H}^1 \llcorner \Gamma\,.$$



# Scaling properties of the energy

**Energies:** induced by the misfit

$$E_{\alpha,r}(\emptyset) := \inf \left\{ \int_{\Omega_r^+} W(\beta)\, dx : \ \operatorname{Curl} \beta = 0 \right\} \qquad \text{(Elastic energy)}$$

$$E_{\alpha,r} := \min_{(\Gamma,B)\in\mathcal{AD}} \inf \left\{ \int_{\Omega_r^+} W(\beta)\, dx : \beta \in \mathcal{AS}(\Gamma, B) \right\} \qquad \text{(Plastic energy)}$$

## Theorem (F., Palombaro, Ponsiglione '15)

*The dislocation-free elastic energy scales like $r^3$: we have $E_{\alpha,1}(\emptyset) > 0$ and*

$$E_{\alpha,r}(\emptyset) = r^3\, E_{\alpha,1}(\emptyset)\,.$$

*The plastic energy scales like $r^2$: there exists $0 < E_\alpha < +\infty$ such that*

$$E_{\alpha,r} = r^2\, E_\alpha + o(r^2)\,.$$

*Large $r \implies$ dislocations are energetically favourable.*

Müller, Palombaro. Calculus of Variations and Partial Differential Equations (2008, 2013).

# Upper bound construction

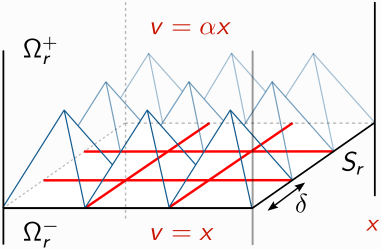
**Goal:** define a square array of edge dislocations with spacing $\delta := \dfrac{b}{\alpha - 1}$.

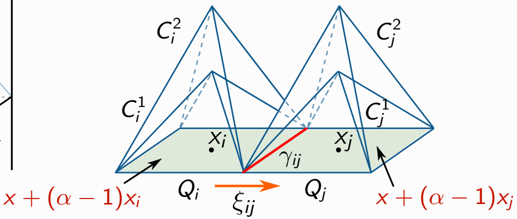
- Divide $S_r$ into $(r/\delta)^2$ squares of side $\delta$.
- Above each $Q_i$ define pyramids $C_i^1$ (height $\delta/2$) and $C_i^2$ (height $\delta$).
- Define deformation $v \in SBV(\Omega_r; \mathbb{R}^3)$, and strain $\beta := \nabla v$ (a.c. part of $Dv$).

**Induced dislocations:** $\operatorname{Curl} \beta = -\sum_{i,j} \xi_{ij} \otimes \dot\gamma_{ij} \, d\mathcal{H}^1 \llcorner \gamma_{ij}$ with

- $\gamma_{ij} := Q_i \cap Q_j$ admissible dislocation curve ($\alpha = 1 + 1/n \implies \delta = nb$)
- $\xi_{ij} := (\alpha - 1)(x_j - x_i) \in \pm b\{e_1, e_2\}$ Burgers vector

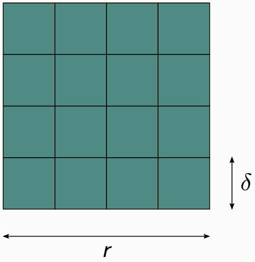
**Energy:** in each pyramid is $c = c(\alpha, b, p) \implies E_{\alpha,r} \leq c\dfrac{r^2}{\delta^2}$ (as $W(\alpha I) = 0$).



# Remarks on the semi-discrete model

**Deformed configuration:** $v(S_R)$ with $v$ from the upper bound construction



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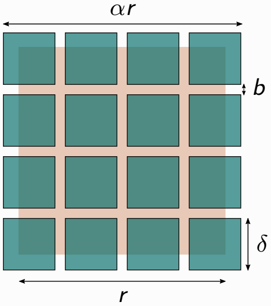

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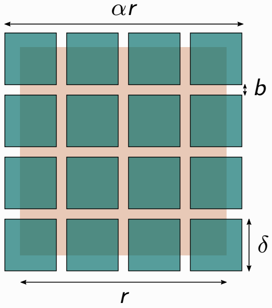


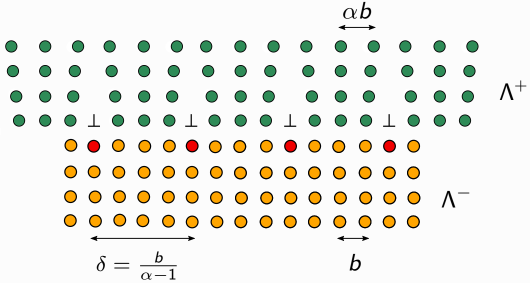

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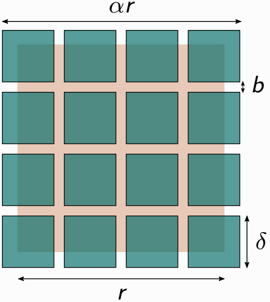
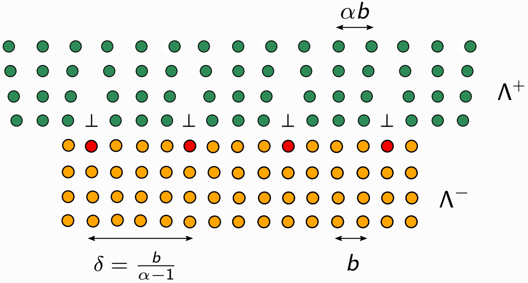


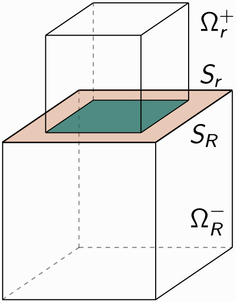

**Limitations of the considered model:**

- $v(S_r)$ does not match $S_r \implies$ not appropriate for semi-coherent interfaces,
- expected dislocation geometry with spacing $\frac{b}{\alpha-1}$ is only optimal in scaling.

**What we do now:**

- take a smaller overlayer and enforce match at the interface,
- introduce a simplified continuum (dislocation density) model to better describe true minimisers.

# Heuristic for the continuum model




**Reference configuration:** $\Omega_{R,r} := \Omega_R^- \cup S_r \cup \Omega_r^+$, with $r := \theta R,\ \theta \in [\alpha^{-1}, 1]$

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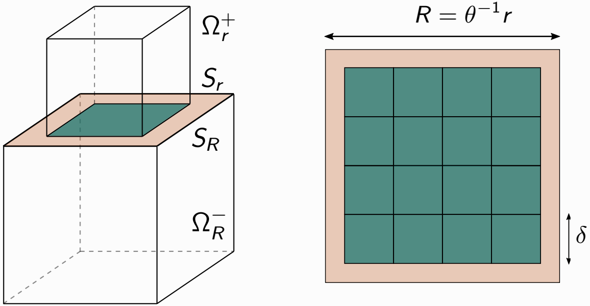


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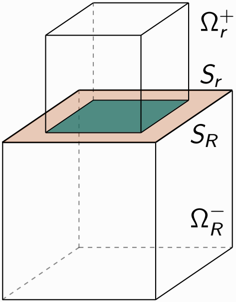


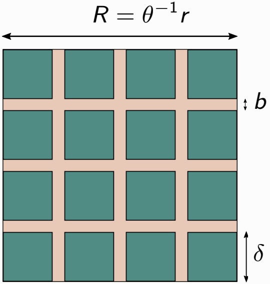


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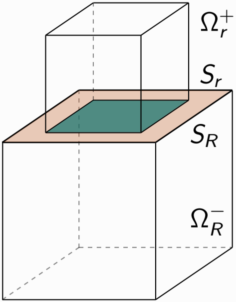


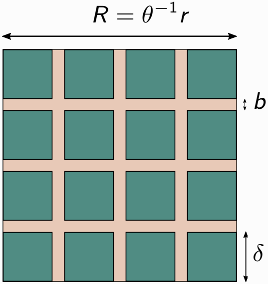


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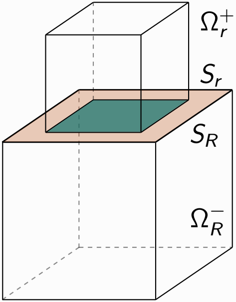
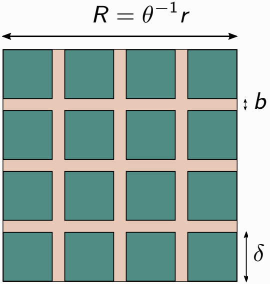
**Upper bound construction:** with $\theta = \alpha^{-1}$ and $\delta = \frac{b}{\theta^{-1}-1} \implies$ perfect match

$$L = 2R\frac{r}{\delta} = \frac{2r^2}{b}(\theta^{-2} - \theta^{-1}) \overset{(\theta^{-1}\approx 1)}{\approx} \frac{r^2}{b}(\theta^{-2} - 1) = \frac{1}{b}(R^2 - r^2) = \frac{1}{b}\,\text{Area Gap}$$

# Heuristic for the continuum model

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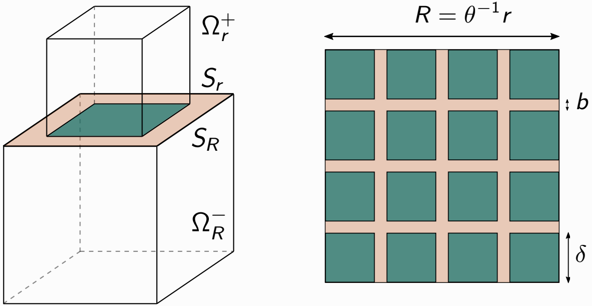
$$\text{Dislocation Length} \approx \frac{1}{b}\,\text{Area Gap}$$

# Heuristic for the continuum model

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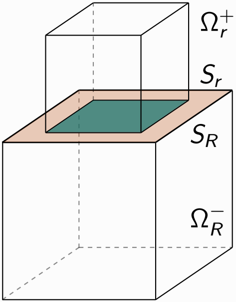
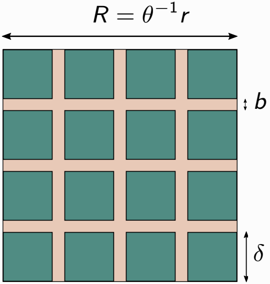
$$\text{Dislocation Length} \approx \frac{1}{b}\,\text{Area Gap}$$

$E_{\alpha,r} \approx r^2 E_\alpha$

# Heuristic for the continuum model

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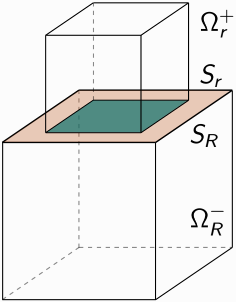
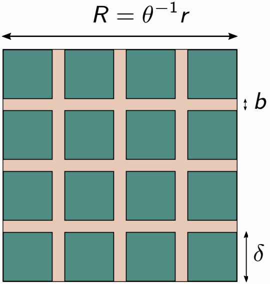
**Upper bound construction:** with $\theta = \alpha^{-1}$ and $\delta = \frac{b}{\theta^{-1}-1}$ $\implies$ perfect match

$$\text{Dislocation Length} \approx \frac{1}{b} \text{Area Gap}$$

$$E_{\alpha,r} \approx r^2 E_\alpha = \sigma \text{ Area Gap} \quad \text{with} \quad \sigma := \frac{E_\alpha}{\theta^{-2} - 1}$$

# Heuristic for the continuum model

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$$\text{Dislocation Length} \approx \frac{1}{b} \text{Area Gap}$$

$$E_{\alpha,r} \approx r^2 E_\alpha = \sigma \text{ Area Gap} \quad \text{with} \quad \sigma := \frac{E_\alpha}{\theta^{-2} - 1}$$

**Hypothesis:** Dislocation Energy $\propto$ Dislocation Length. Then optimise over $\theta$.

# Continuum model

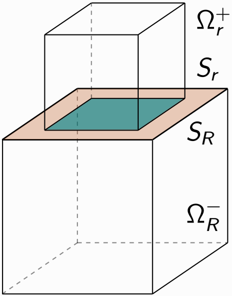
**Reference configuration:** $\Omega_{R,r} := \Omega_R^- \cup S_r \cup \Omega_r^+$, with $r := \theta R\,,\ \theta \in [\alpha^{-1}, 1]$

**Deformations:** $v \in W^{1,2}(\Omega_r^+;\mathbb{R}^3)$ such that $v = \dfrac{x}{\theta}$ on $S_r$

$\implies v(S_r) = S_R$ (interface match)

**Energy density:** $W(F) \sim \text{dist}(F, \alpha SO(3))^2$

**Elastic:** $E^{el}_{\alpha,R}(\theta) := \inf\left\{\int_{\Omega_r^+} W(\nabla v)\,dx :\ v = x/\theta \text{ on } S_r\right\}$



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**Deformations:** $v \in W^{1,2}(\Omega_r^+; \mathbb{R}^3)$ such that $v = \dfrac{x}{\theta}$ on $S_r$

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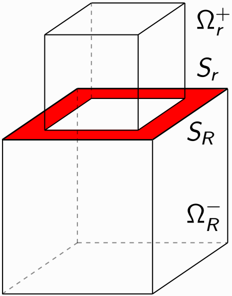

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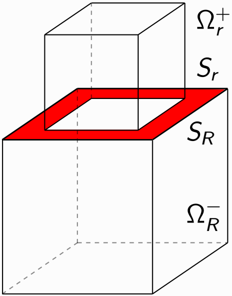


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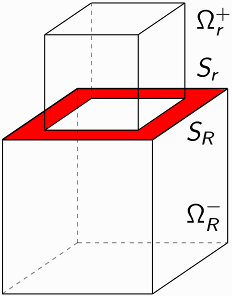


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**Question:** behaviour of $E_{\alpha,R}^{tot}(\theta)$ as $R \to \infty$ ?

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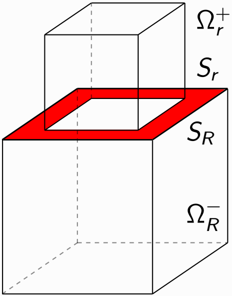


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**Question:** behaviour of $E^{tot}_{\alpha,R}(\theta)$ as $R \to \infty$ ?

**Energy competition:**

- $\theta = 1 \implies$ no dislocation energy
- $\theta = \alpha^{-1} \implies$ no elastic energy $\qquad (v := \alpha x,\ W(\alpha I) = 0)$
- $\theta \in (\alpha^{-1}, 1) \implies$ both present

# Asymptotic for $E^{tot}_{\alpha,R}$

Let $\theta_R \in [\alpha^{-1}, 1]$ be a minimiser for $E^{tot}_{\alpha,R}$ and define

$$\mathcal{E}^{el}(R) := \frac{\sigma^2}{\alpha^3 C^{el}}\, R\,, \qquad \mathcal{E}^{pl}(R) := \sigma R^2 \left(1 - \frac{1}{\alpha^2}\right) - 2\frac{\sigma^2}{\alpha^3 C^{el}} R\,.$$

## Theorem (F., Palombaro, Ponsiglione '15)

*As $R \to +\infty$ we have*

$$E^{el}_{\alpha,R}(\theta_R) = \mathcal{E}^{el}(R) + O(R)\,, \qquad E^{pl}_R(\theta_R) = \mathcal{E}^{pl}(R) + O(R)\,,$$

*and therefore*

$$E^{tot}_{\alpha,R} = \mathcal{E}^{el}(R) + \mathcal{E}^{pl}(R) + o(R).$$

*In particular, for large $R$:*

- *dislocations are energetically more favourable,*
- *dislocation spacing (density) tends to $\delta = \frac{b}{\alpha - 1}$,*
- *far field stress is relieved.*

# Idea of the Proof

**Step 1.** Rescale the elastic energy

$$E^{el}_{\alpha,R}(\theta) = R^3\theta^3\, E^{el}_{\alpha,1}(\theta)$$

**Step 2.** Let $\theta_R \in [\alpha^{-1}, 1]$ be a minimiser of $E^{tot}_{\alpha,R}$. Then, as $R \to \infty$

$$E^{el}_{\alpha,1}(\theta_R) \to 0\,, \quad \theta_R \to \alpha^{-1} \implies \text{Linearisation (about } \alpha I)$$

**Step 3.** There exists $C^{el} > 0$ such that, as $R \to \infty$,

$$\frac{1}{(\theta_R^{-1} - \alpha)^2} E^{el}_{\alpha,1}(\theta_R) \to C^{el}$$

**Step 4.** Write the elastic energy as a polynomial

$$E^{el}_{\alpha,R}(\theta_R) = R^3\theta_R^3(\theta_R^{-1} - \alpha)^2\, \frac{1}{(\theta_R^{-1} - \alpha)^2} E^{el}_{\alpha,1}(\theta_R) = k^{el}_R R^3\theta_R^3(\theta_R^{-1} - \alpha)^2$$

where $k^{el}_R := C^{el} + \varepsilon_R > 0$ and $k^{el}_R \to C^{el}$.

Dal Maso, Negri, Percivale. Set-Valued Analysis (2002).

# Idea of the Proof

**Step 5.** The total energy computed along $\theta_R$ is equal to

$$E^{tot}_{\alpha,R}(\theta_R) = k^{el}_R\, R^3 \theta_R^3 (\theta_R^{-1} - \alpha)^2 + \sigma R^2 (1 - \theta_R^2) \qquad (1.1)$$

with $\theta_R \to \alpha^{-1}$ minimisers and $k^{el}_R \to C^{el}$.

**Step 6.** For a fixed parameter $k > 0$, introduce the family of polynomials

$$P_{R,k}(\theta) := k\, R^3 \theta^3 (\theta^{-1} - \alpha)^2 + \sigma R^2 (1 - \theta^2)$$

**Step 7.** Show that for $R \gg 0$ there exists a unique minimiser $\theta^m_R$ to

$$P_{R,k}(\theta^m_R) = \min_{\theta \in [\alpha^{-1}, 1]} P_{R,k}(\theta)\,.$$

Moreover $\theta^m_R \to \alpha^{-1}$.

**Step 8.** Since $\theta_R - \theta^m_R \to 0$, by using (1.1), minimality, and computing $P_{R,k}(\theta^m_R)$, we have the thesis

$$E^{tot}_{\alpha,R}(\theta_R) = \underbrace{\frac{\sigma^2}{\alpha^3 C^{el}}\, R}_{\text{Elastic}} + \underbrace{\sigma R^2 \left(1 - \alpha^{-2}\right) - 2 \frac{\sigma^2}{\alpha^3 C^{el}}\, R}_{\text{Plastic}} + O(R)\,.$$

# Conclusions and Perspectives

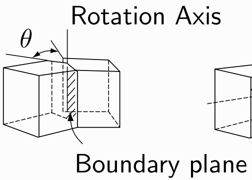
## Conclusions:

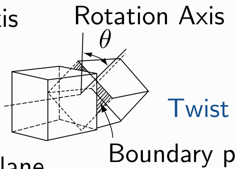
- A basic variational model describing the competition between the plastic energy spent at interfaces, and the corresponding release of bulk energy.
- The size of the overlayer is a free parameter $\Longrightarrow$ free boundary problem, but only through the scalar parameter $\theta$.

## Perspectives:

- Grain boundaries, the misfit between the crystal lattices are described by rotations rather than dilations.
  Read, Shockley (1950) - Hirth, Carnahan (1992)
- Optimal geometry for the dislocation net (square vs hexagonal)
  Koslowski, Ortiz (2004)

Tilt boundary




Twist boundary

# Presentation Plan

## 1 Geometric Patterns of Dislocations

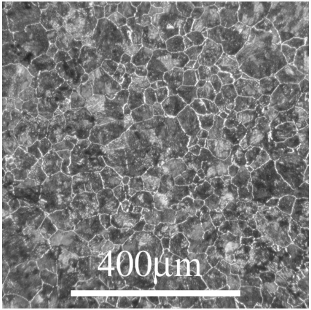
- Dislocations
- Semi-coherent interfaces
- Linearised polycrystals

## 2 Microgeometries in Composites

- Critical lower integrability
- Convex integration
- Proof of our main result

# Polycrystals

**Polycrystal:** formed by many grains, having the **same** lattice structure, mutually rotated $\implies$ interface misfit at **grain boundaries**.



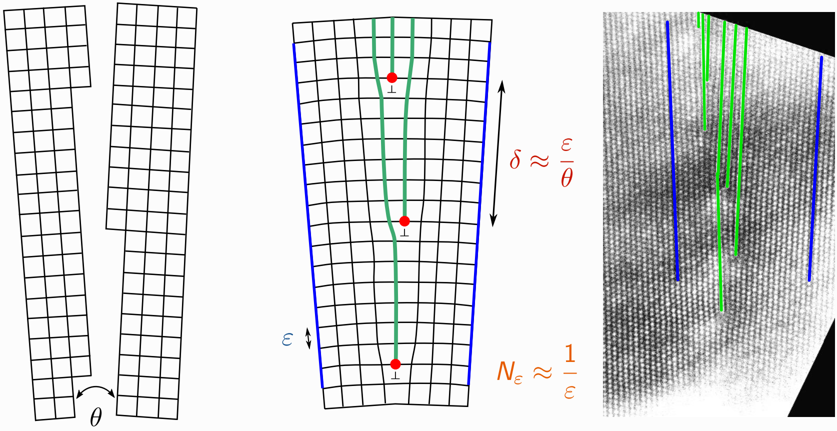

**Goal:** obtain polycrystalline structures as minimisers of some energy functional.

F., Palombaro, Ponsiglione. *Linearised Polycrystals from a 2D System of Edge Dislocations.* Preprint (2017)

# Tilt grain boundaries

**Tilt boundary:** small angle rotation $\theta$ between grains $\implies$ **edge dislocations**.

**Boundary structure:** periodic array of edge dislocations with spacing $\boldsymbol{\delta = \frac{\varepsilon}{\theta}}$.




Porter, Easterling. CRC Press (2009) - Gottstein. Springer (2013)

# Plan

**Setting:** consider a 2D system of $N_\varepsilon$ edge dislocations, where $\varepsilon > 0$ is the lattice spacing and

$$N_\varepsilon \to +\infty \quad \text{as} \quad \varepsilon \to 0 \,.$$

Let $\mathcal{F}_\varepsilon$ be the energy of such system.

**Plan:**

- compute $\mathcal{F}$, the Γ-limit of $\mathcal{F}_\varepsilon$ as $\varepsilon \to 0$,
- show that under suitable boundary conditions $\mathcal{F}$ is minimised by polycrystals.

**Linearised polycrystals:** our energy regime will imply

$$N_\varepsilon \ll \frac{1}{\varepsilon}$$

$\implies$ we have less dislocations than tilt grain boundaries. However we still obtain polycrystalline minimisers, but with grains rotated by an infinitesimal angle $\theta \approx 0$.

# Setting (linearised planar elasticity)

**Reference configuration:** $\Omega \subset \mathbb{R}^2$ open bounded.

**Dislocation lines:** points $x_0 \in \Omega$ separated by $2\varepsilon$.

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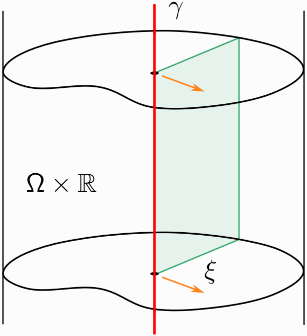


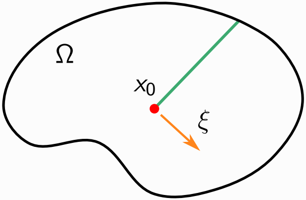

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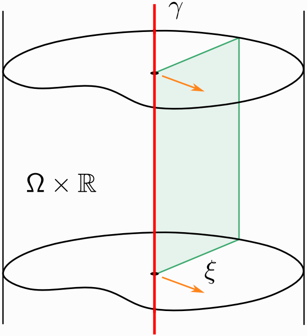


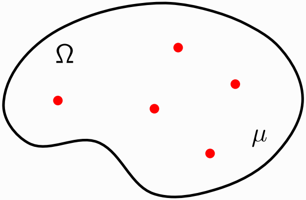

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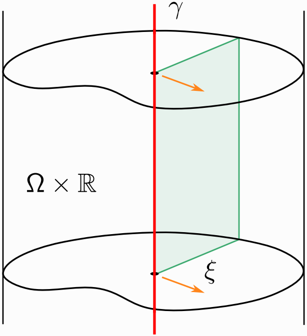


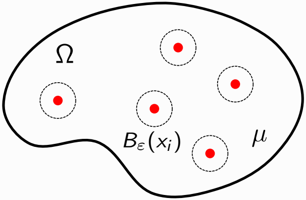

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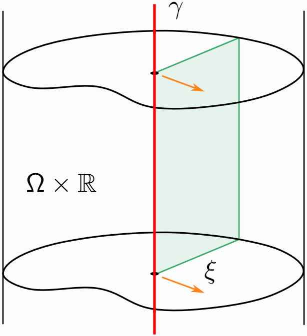


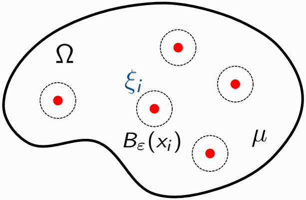

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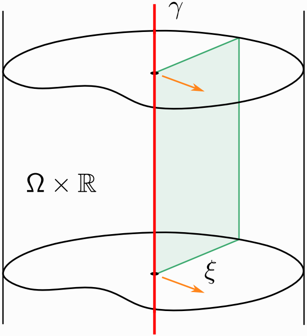


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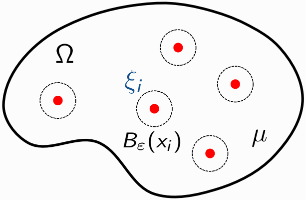



**Linearised Energy:** $\mathbb{C}F : F \sim |F^{\mathrm{sym}}|^2$, then

$$E_\varepsilon(\mu, \beta) := \int_{\Omega_\varepsilon(\mu)} \mathbb{C}\beta : \beta \, dx = \int_\Omega \mathbb{C}\beta : \beta \, dx \,.$$

# Self-energy of a single dislocation core

Let $\beta$ generate $\xi\,\delta_0$, that is "$\mathrm{Curl}\,\beta = \xi\,\delta_0$"

$$\int_{B_1\setminus B_\varepsilon} |\beta|^2\,dx \geq \int_\varepsilon^1 \int_{\partial B_\rho} |\beta\cdot t|^2\,ds\,d\rho \geq \text{(Jensen)}$$

$$\geq \frac{1}{2\pi}\int_\varepsilon^1 \frac{1}{\rho}\left|\int_{\partial B_\rho} \beta\cdot t\,ds\right|^2 d\rho = \frac{|\xi|^2}{2\pi}|\log\varepsilon|\,.$$

The reverse inequality can be obtained by computing the energy of

$$\beta(x) := \frac{1}{2\pi}\,\xi\otimes J\frac{x}{|x|^2}\,,\quad J := \text{clock-wise rotation of } \frac{\pi}{2}\,.$$

**Remark:** let $s\in(0,1)$, then

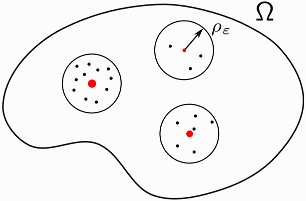
$$\int_{B_{\varepsilon^s}\setminus B_\varepsilon} |\beta|^2\,dx \geq (1-s)\frac{|\xi|^2}{2\pi}|\log\varepsilon|$$

**Self-energy:** is of order $|\log\varepsilon|$ and concentrated in a small region around $B_\varepsilon$.

# The Hard Core assumption

**HC Radius:** fixed scale $\rho_\varepsilon \gg \varepsilon$ .

**Clusters** of dislocations at scale $\rho_\varepsilon$ are identified with a single **multiple dislocation**.



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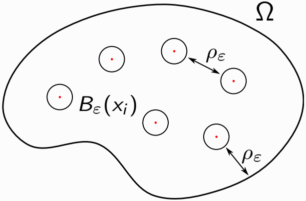
with $\mathbb{S} := \mathrm{Span}_{\mathbb{Z}} \, \mathcal{S}$ set of multiple Burgers vectors, and

$$|x_i - x_j| > 2\rho_\varepsilon \, , \quad \mathrm{dist}(x_k, \partial\Omega) > \rho_\varepsilon \, .$$



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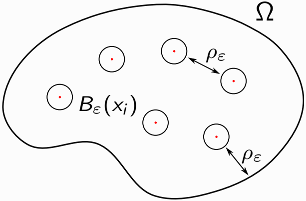



**Hypothesis on HC Radius:** as $\varepsilon \to 0$

- $\rho_\varepsilon / \varepsilon^s \to \infty, \ \forall \, s \in (0,1)$ , (HC contains almost all the self-energy)
- $N_\varepsilon \rho_\varepsilon^2 \to 0$ . (Measure of HC region vanishes)

# Energy regimes

**Energy scaling:** each dislocation accounts for $|\log \varepsilon| \implies$ relevant scaling is

$$E_\varepsilon \approx N_\varepsilon |\log \varepsilon| \,,$$

**Rescaled energy functionals:**

$$\mathcal{F}_\varepsilon(\mu, \beta) := \frac{1}{N_\varepsilon |\log \varepsilon|} \int_{\Omega_\varepsilon(\mu)} \mathbb{C}\beta : \beta \, dx \,.$$

**Energy regimes:** the behaviour of $N_\varepsilon$ determines three different regimes:

- $N_\varepsilon \ll |\log \varepsilon| \rightsquigarrow$ Dilute dislocations
- $N_\varepsilon \approx |\log \varepsilon| \rightsquigarrow$ Critical regime

  Garroni, Leoni, Ponsiglione. *Gradient theory for plasticity via homogenization of discrete dislocations.* J. Eur. Math. Soc. (JEMS) (2010)

- $N_\varepsilon \gg |\log \varepsilon| \rightsquigarrow$ Super-critical regime

  F., Palombaro, Ponsiglione. *Linearised Polycrystals from a 2D System of Edge Dislocations.* Preprint (2017)

# Candidate Γ-limit

Let $(\mu,\beta)$ with $\mu=\sum_{i=1}^N \xi_i\,\delta_{x_i}$ be such that "Curl $\beta=\mu$".

**Energy decomposition:** let $\mathrm{HC}_\varepsilon(\mu):=\cup_{i=1}^N B_{\rho_\varepsilon}(x_i)$ be the HC region

$$E_\varepsilon(\mu,\beta)=\int_{\Omega\setminus \mathrm{HC}_\varepsilon(\mu)}\mathbb{C}\beta:\beta\,dx+\int_{\mathrm{HC}_\varepsilon(\mu)}\mathbb{C}\beta:\beta\,dx\,.$$

**Γ-limit:** $S\in L^2(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$, $A\in L^2(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{skew}})$, $\mu\in\mathcal{M}(\Omega;\mathbb{R}^2)$ with Curl $A=\mu$,

$$\mathcal{F}(\mu,S,A):=\int_\Omega \mathbb{C}S:S\,dx+\int_\Omega\varphi\left(\frac{d\mu}{d|\mu|}\right)\,d|\mu|\,.$$

**Density $\varphi$:** the self-energy for a single dislocation core $\xi\delta_0$ is

$$\psi(\xi):=\lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\min_\beta\left\{\int_{B_1\setminus B_\varepsilon}\mathbb{C}\beta:\beta\,dx:\ \text{"Curl }\beta=\xi\delta_0\text{"}\right\}\,.$$

Define $\varphi\colon\mathbb{R}^2\to[0,\infty)$ as the relaxation of $\psi$ (splitting multiple dislocations)

$$\varphi(\xi):=\min\left\{\sum_{i=1}^M\lambda_i\psi(\xi_i):\ \xi=\sum_{i=1}^M\lambda_i\xi_i,\ M\in\mathbb{N},\ \lambda_i\geq 0,\ \xi_i\in\mathbb{S}\right\}\,.$$

# $\Gamma$-convergence result for $N_\varepsilon \gg |\log \varepsilon|$

## Theorem (F., Palombaro, Ponsiglione '17)

**Compactness:** *consider* $(\mu_\varepsilon, \beta_\varepsilon)$ *s.t.* "$\operatorname{Curl} \beta_\varepsilon = \mu_\varepsilon$" *and* $\mathcal{F}_\varepsilon(\mu_\varepsilon, \beta_\varepsilon) \leq C \implies$

- $\dfrac{\beta_\varepsilon^{\mathrm{sym}}}{\sqrt{N_\varepsilon |\log \varepsilon|}} \rightharpoonup S\,, \quad \dfrac{\beta_\varepsilon^{\mathrm{skew}}}{N_\varepsilon} \rightharpoonup A$ *in* $L^2(\Omega; \mathbb{M}^{2\times 2})$,
- $\dfrac{\mu_\varepsilon}{N_\varepsilon} \overset{*}{\rightharpoonup} \mu$ *in* $\mathcal{M}(\Omega; \mathbb{R}^2)$,
- $\mu \in H^{-1}(\Omega; \mathbb{R}^2)$ *and* $\operatorname{Curl} A = \mu$.

**$\Gamma$-convergence:** *the functionals* $\mathcal{F}_\varepsilon$ $\Gamma$*-converge to*

$$\mathcal{F}(\mu, S, A) := \int_\Omega \mathbb{C}S : S\, dx + \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu|\,, \quad \textit{with} \quad \operatorname{Curl} A = \mu\,.$$

## Remark:

- $S$ and $A$ live on two different scales with $S \ll A \implies$ terms in $\mathcal{F}$ decoupled.
- In the critical regime $N_\varepsilon \approx |\log \varepsilon|$ we have $S \approx A$ and $\operatorname{Curl}(S + A) = \mu$.

# Compactness of the measures

Let $\mu_n := \sum_{i=1}^{M_n} \xi_{n,i}\delta_{x_{n,i}}$ and "Curl $\beta_n = \mu_n$". We show that

$$\frac{1}{N_{\varepsilon_n}}|\mu_n|(\Omega) = \frac{1}{N_{\varepsilon_n}}\sum_{i=1}^{M_n}|\xi_{n,i}| \leq C\,, \tag{1.2}$$

so that $\frac{\mu_n}{N_{\varepsilon_n}} \overset{*}{\rightharpoonup} \nu$.

$$C \geq \mathcal{F}_{\varepsilon_n}(\mu_n,\beta_n) \geq \frac{1}{N_{\varepsilon_n}}\sum_{i=1}^{M_n}\frac{1}{|\log\varepsilon_n|}\int_{B_{\rho_{\varepsilon_n}}(x_{n,i})\setminus B_{\varepsilon_n(x_{n,i})}} W(\beta_n)\,dx$$

$$\geq \frac{1}{N_{\varepsilon_n}}\sum_{i=1}^{M_n}\psi_{\varepsilon_n}(\xi_{n,i}) = \frac{1}{N_{\varepsilon_n}}\sum_{i=1}^{M_n}|\xi_{n,i}|^2\,\psi_{\varepsilon_n}\left(\frac{\xi_{n,i}}{|\xi_{n,i}|}\right) \geq \frac{c}{N_{\varepsilon_n}}\sum_{i=1}^{M_n}|\xi_{n,i}|^2$$

$$\geq \frac{c}{N_{\varepsilon_n}}\sum_{i=1}^{M_n}|\xi_{n,i}| = c\,\frac{|\mu_n|(\Omega)}{N_{\varepsilon_n}} \implies (1.2)$$

# Compactness of the strains

**Symmetric Part:**

$$CN_{\varepsilon_n}|\log \varepsilon_n| \geq CE_{\varepsilon_n}(\mu_n, \beta_n) \geq C\int_\Omega |\beta_n^{\rm sym}|^2\, dx \implies \frac{\beta_n^{\rm sym}}{\sqrt{N_{\varepsilon_n}|\log \varepsilon_n|}} \rightharpoonup S\,.$$

**Skew Part:** since "$\operatorname{Curl}\beta_n = \mu_n$" we can apply the generalised Korn inequality:

$$\int_\Omega |\beta_n^{\rm skew}|^2\, dx \leq C\left(\int_\Omega |\beta_n^{\rm sym}|^2\, dx + \left(|\mu_n|(\Omega)\right)^2\right) \qquad \text{(Gen. Korn)}$$

$$\leq C\left(\sqrt{N_{\varepsilon_n}|\log \varepsilon_n|} + N_{\varepsilon_n}^2\right) \leq CN_{\varepsilon_n}^2 \qquad (N_\varepsilon \gg |\log \varepsilon|)$$

so that $\dfrac{\beta_n^{\rm skew}}{N_{\varepsilon_n}} \rightharpoonup A$.

Garroni, Leoni, Ponsiglione. *Gradient theory for plasticity via homogenization of discrete dislocations.* J. Eur. Math. Soc. (JEMS) (2010)

# Adding boundary conditions

**Dirichlet type BC:** at level $\varepsilon > 0$ fix a boundary condition $g_\varepsilon \colon \Omega \to \mathbb{M}^{2\times 2}$ s.t.

$$\frac{g_\varepsilon^{\mathrm{sym}}}{\sqrt{N_\varepsilon |\log \varepsilon|}} \rightharpoonup g_S \,, \qquad \frac{g_\varepsilon^{\mathrm{skew}}}{N_\varepsilon} \rightharpoonup g_A \,.$$

**Admissible dislocations:** measures $\mu$ satisfying

$$\mu(\Omega) = \int_{\partial\Omega} g_\varepsilon \cdot t \, ds \,. \tag{GND}$$

**Admissible strains:** $\beta \colon \Omega_\varepsilon(\mu) \to \mathbb{M}^{2\times 2}$ such that " $\mathrm{Curl}\, \beta = \mu$" and

$$\beta \cdot t = g_\varepsilon \cdot t \quad \text{on} \quad \partial\Omega \,.$$

**$\Gamma$-limit:** the usual energy $\mathcal{F}_\varepsilon$ Γ-converges to

$$\mathcal{F}_{\mathrm{BC}}(\mu, S, A) := \int_\Omega \mathbb{C} S : S \, dx + \int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right) d|\mu| + \int_{\partial\Omega} \varphi((g_A - A)\cdot t)\, ds \,,$$

such that $\mathrm{Curl}\, A = \mu$, with $\mu \in \mathcal{M}(\Omega; \mathbb{R}^2) \cap H^{-1}(\Omega; \mathbb{R}^2)$.

**Remark:** $\beta_\varepsilon^{\mathrm{sym}} \ll \beta_\varepsilon^{\mathrm{skew}} \implies$ BC pass to the limit only for $A$.

# Minimising $\mathcal{F}_{\mathrm{BC}}$ with piecewise constant BC

**Remark:** there are no BC on $S \implies$ we can neglect elastic energy.

**Piecewise constant BC:** Fix a piecewise constant BC

$$g_A := \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad a := \sum_{k=1}^{M} m_k \, \chi_{U_k} \,,$$

with $m_k < m_{k+1}$ and $\{U_k\}_{k=1}^M$ Caccioppoli partition of $\Omega$.

## Problem

*Minimise*

$$\mathcal{F}_{\mathrm{BC}}(\mu, 0, A) = \int_\Omega \varphi \left( \frac{d\mu}{d|\mu|} \right) \, d|\mu| + \int_{\partial\Omega} \varphi((g_A - A) \cdot t) \, ds \,,$$

*with* $\operatorname{Curl} A = \mu$ *and* $\mu \in \mathcal{M}(\Omega; \mathbb{R}^2) \cap H^{-1}(\Omega; \mathbb{R}^2)$.

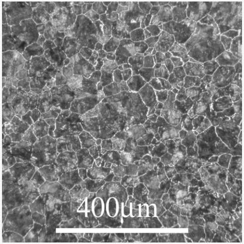
# Polycrystals as energy minimisers

## Theorem (F., Palombaro, Ponsiglione '17)

*Given a piecewise constant boundary condition $g_A$, there exists a piecewise constant minimiser of $\mathcal{F}_{\mathrm{BC}}(\mu, 0, A)$*

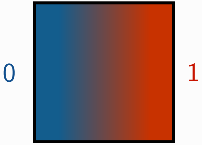
$$A = \sum_{k=1}^{M} A_k \chi_{E_k} \,,$$

*with $A_k \in \mathbb{M}^{2\times 2}_{\mathrm{skew}}$ and $\{E_k\}_{k=1}^M$ Caccioppoli partition of $\Omega$. We interpret $A$ as a linearised polycrystal.*




**Open Question:** Are all minimisers piecewise constant? Uniqueness?

**Essential:** that the boundary condition is piecewise affine on the whole $\partial\Omega$.

# Idea of the proof

**Problem:** given a piecewise constant BC $g_A$, consider

$$\inf\left\{\int_\Omega \varphi\left(\frac{d\mu}{d|\mu|}\right)\,d|\mu| + \int_{\partial\Omega}\varphi((g_A - A)\cdot t)\,ds :\ \operatorname{Curl} A = \mu \in \mathcal{M}\cap H^{-1}\right\}.$$

Since $A$ and $g_A$ are antisymmetric, $\exists\ u, a \in L^2(\Omega)$ s.t.

$$A = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}, \quad g_A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}.$$

**Note:** $\operatorname{Curl} A = Du \in \mathcal{M}(\Omega;\mathbb{R}^2) \implies u \in BV(\Omega) \implies$ **Equivalent Problem**:

$$\inf\left\{\int_\Omega \varphi\left(\frac{dDu}{d|Du|}\right)\,d|Du| + \int_{\partial\Omega}\varphi((u-a)\nu)\,ds :\ u \in BV(\Omega)\right\}. \qquad (1.3)$$

**Proof:** let $\tilde{u}$ be a minimiser for (1.3). By anisotropic Coarea Formula

$$\int_\Omega \varphi\left(\frac{dD\tilde{u}}{d|D\tilde{u}|}\right)\,d|D\tilde{u}| = \int_{\mathbb{R}} Per_\varphi(\{x\in\Omega :\ \tilde{u}(x) > t\})\,dt\,,$$

we can select the levels with minimal perimeter. This defines the Caccioppoli partition.

# Comparison with classical Read-Shockley formula

**Read-Shockley formula:** Elastic energy$= E_0\theta(1+|\log\theta|)$.

- This energy corresponds to small rotations $\theta$ between grains: small rotations but larger than linearised rotations.
- It is a nonlinear formula that corresponds to a higher energy regime.
- The density of dislocations to obtain small rotations is

$$\text{Density} \approx \frac{1}{\varepsilon} \gg N_\varepsilon \,.$$

**Question:** Γ-convergence analysis of the Read-Shockley formula?

Lauteri, Luckhaus. *An energy estimate for dislocation configurations and the emergence of Cosserat-type structures in metal plasticity.* Preprint (2017)

**Question:** Are there some relevant energy regimes in between?

# Conclusions and Perspectives

**Conclusions:**

- A variational model for linearised polycrystals with infinitesimal rotations between the grains, deduced by Γ-convergence.
- Networks of dislocations are obtained as the result of energy minimisation, under suitable boundary conditions.

**Perspectives:**

- Uniqueness of piecewise constant minimisers?
- Comparison with the Read-Shockley formula?
  Lauteri, Luckhaus. Preprint (2017).
- Dynamics for linearised polycrystals?
  Taylor. *Crystalline variational problems*. Bull. Amer. Math. Soc. (1978).
  Chambolle, Morini, Ponsiglione. *Existence and Uniqueness for a Crystalline Mean Curvature Flow.* Comm. Pure Appl. Math (2017).
- Supercritical regime analysis starting from a non-linear energy?
  Müller, Scardia, Zeppieri. *Geometric rigidity for incompatible fields and an application to strain-gradient plasticity.* Indiana University Mathematics Journal (2014).

# Presentation Plan

1. **Geometric Patterns of Dislocations**
   - Dislocations
   - Semi-coherent interfaces
   - Linearised polycrystals

2. **Microgeometries in Composites**
   - Critical lower integrability
   - Convex integration
   - Proof of our main result

# Gradient integrability for solutions to elliptic equations

$\Omega \subset \mathbb{R}^2$ bounded open domain. A map $\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2})$ is **uniformly elliptic** if

$$\sigma\xi \cdot \xi \geq \lambda|\xi|^2 , \qquad \forall \xi \in \mathbb{R}^2, x \in \Omega .$$

## Problem

*Study the gradient integrability of distributional solutions $u \in W^{1,1}(\Omega)$ to*

$$\operatorname{div}(\sigma \nabla u) = 0 , \tag{2.1}$$

*when*

$$\sigma = \sigma_1 \chi_{E_1} + \sigma_2 \chi_{E_2} ,$$

*with $\sigma_1, \sigma_2 \in \mathbb{M}^{2\times 2}$ constant elliptic matrices, $\{E_1, E_2\}$ measurable partition of $\Omega$.*

**Application to composites:**

- $\Omega$ is a section of a composite conductor obtained by mixing two materials with conductivities $\sigma_1$ and $\sigma_2$,
- the electric field $\nabla u$ solves (2.1),
- concentration of $\nabla u$ in relation to the geometry $\{E_1, E_2\}$.

# Astala's Theorem


## Theorem (Astala '94)

*Let* $\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2})$ *be uniformly elliptic. There exists exponents* $1 < q < 2 < p$ *such that if* $u \in W^{1,q}(\Omega)$ *solves*

$$\operatorname{div}(\sigma \nabla u) = 0\,,$$

*then* $\nabla u \in L^p_{\mathrm{weak}}(\Omega; \mathbb{R}^2)$.

## Question

*Are the exponents* $q$ *and* $p$ *optimal among two-phase elliptic conductivities*

$$\sigma = \sigma_1 \chi_{E_1} + \sigma_2 \chi_{E_2}\,?$$

Astala. *Area distortion of quasiconformal mappings.* Acta Mathematica (1994)

# Astala's exponents for two-phase conductivities


For two-phase conductivities Astala's exponents $q = q_{\sigma_1,\sigma_2}$ and $p = p_{\sigma_1,\sigma_2}$ have been characterised.

**Remark:** it is sufficient to prove optimality in the case

$$\sigma_1 = \begin{pmatrix} 1/K & 0 \\ 0 & 1/S_1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} K & 0 \\ 0 & S_2 \end{pmatrix},$$

where

$$K > 1 \qquad \text{and} \qquad \frac{1}{K} \leq S_j \leq K\,, \quad j = 1, 2\,.$$

The corresponding critical exponents for Astala's theorem are

$$q_{\sigma_1,\sigma_2} = \frac{2K}{K+1}, \quad p_{\sigma_1,\sigma_2} = \frac{2K}{K-1}.$$

Nesi, Palombaro, Ponsiglione. Ann. Inst. H. Poincaré Anal. Non Linéaire (2014).

# Upper exponent optimality


## Theorem (Nesi, Palombaro, Ponsiglione '14)

*Let* $\sigma_1 = \mathrm{diag}(1/K, 1/S_1), \sigma_2 = \mathrm{diag}(K, S_2)$ *with* $K > 1$ *and* $S_1, S_2 \in [1/K, K]$.

- (i) *If* $\sigma \in L^\infty(\Omega; \{\sigma_1, \sigma_2\})$ *and* $u \in W^{1,\frac{2K}{K+1}}(\Omega)$ *solves*

$$\mathrm{div}(\sigma \nabla u) = 0 \tag{2.2}$$

  *then* $\nabla u \in L^{\frac{2K}{K-1}}_{\mathrm{weak}}(\Omega; \mathbb{R}^2)$.

- (ii) *There exists* $\bar{\sigma} \in L^\infty(\Omega; \{\sigma_1, \sigma_2\})$ *and a weak solution* $\bar{u} \in W^{1,2}(\Omega)$ *to* (2.2) *with* $\sigma = \bar{\sigma}$, *satisfying affine boundary conditions and such that* $\nabla \bar{u} \notin L^{\frac{2K}{K-1}}(\Omega; \mathbb{R}^2)$.

## Question we address

Is the lower exponent $\frac{2K}{K+1}$ optimal?

# Lower exponent optimality


## Theorem (F., Palombaro '17)

*Let* $\sigma_1 = \mathrm{diag}(1/K, 1/S_1), \sigma_2 = \mathrm{diag}(K, S_2)$ *with* $K > 1$ *and* $S_1, S_2 \in [1/K, K]$. *There exist*

- *coefficients* $\sigma_n \in L^\infty(\Omega; \{\sigma_1; \sigma_2\})$,
- *exponents* $p_n \in \left[1, \frac{2K}{K+1}\right]$,
- *functions* $u_n \in W^{1,1}(\Omega)$ *such that* $u_n(x) = x_1$ *on* $\partial\Omega$,

*such that*

$$\mathrm{div}(\sigma_n \nabla u_n) = 0\,,$$

$$\nabla u_n \in L^{p_n}_{\mathrm{weak}}(\Omega; \mathbb{R}^2), \quad p_n \to \frac{2K}{K+1}, \quad \nabla u_n \notin L^{\frac{2K}{K+1}}(\Omega; \mathbb{R}^2)\,.$$

F., Palombaro. Calculus of Variations and Partial Differential Equations (2017)

# Presentation Plan

## ❶ Geometric Patterns of Dislocations

- Dislocations
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## ❷ Microgeometries in Composites

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# Solving differential inclusions

## Theorem (Approximate solutions for two phases)

*Let* $A, B \in \mathbb{M}^{2\times 2}$, $C := \lambda A + (1-\lambda)B$ *with* $\lambda \in [0,1]$, *and* $\delta > 0$. *Assume that*

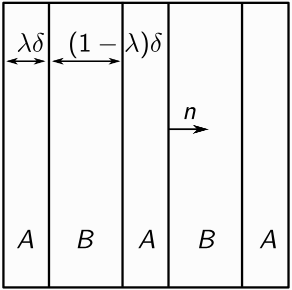
$$B - A = a \otimes n \quad \text{for some} \quad a \in \mathbb{R}^2, n \in S^1 \,. \qquad \text{(Rank-one connection)}$$

$\exists$ **piecewise affine Lipschitz** *map* $f : \Omega \to \mathbb{R}^2$ *such that* $f(x) = Cx$ *on* $\partial\Omega$ *and*

$$\operatorname{dist}(\nabla f, \{A, B\}) < \delta \quad a.e. \text{ in } \quad \Omega \,.$$

**Solutions:** built through **simple laminates**

- rank-one connection allows to laminate in direction $n$,
- $\nabla f$ oscillates in $\delta$-neighbourhoods of $A$ and $B$,
- $\lambda$ proportion for $A$, $1-\lambda$ proportion for $B$,
- this allows to recover boundary data $C$.




Müller. *Variational models for microstructure and phase transitions.*

# Laminates of first order

$\mathcal{L}^2_\Omega$ is the normalised Lebesgue measure restricted to $\Omega \rightsquigarrow \mathcal{L}^2_\Omega(B) := |B \cap \Omega|/|\Omega|$.

## Gradient distribution

Let $f\colon \Omega \to \mathbb{R}^2$ be Lipschitz. The **gradient distribution** of $f$ is the Radon measure $\nabla f_\#(\mathcal{L}^2_\Omega)$ on $\mathbb{M}^{2\times 2}$ defined by

$$\nabla f_\#(\mathcal{L}^2_\Omega)(V) := \mathcal{L}^2_\Omega((\nabla f)^{-1}(V)), \quad \forall \text{ Borel set } V \subset \mathbb{M}^{2\times 2}.$$

Let $f_\delta$ be the map given by the previous Theorem. Then as $\delta \to 0$,

$$\nu_\delta := (\nabla f_\delta)_\#(\mathcal{L}^2_\Omega) \overset{*}{\rightharpoonup} \nu := \lambda\delta_A + (1-\lambda)\delta_B \quad \text{in} \quad \mathcal{M}(\mathbb{M}^{2\times 2}).$$

The measure $\nu$ is called a **laminate of first order**, and it encodes:

- **Oscillations** of $\nabla f_\delta$ about $\{A, B\}$ and their proportions.
- **Boundary condition** since the barycentre of $\nu$ is $\overline{\nu} := \int_{\mathbb{M}^{2\times 2}} M \, d\nu(M) = C$.
- **Integrability** since for $p > 1$ we have

$$\frac{1}{|\Omega|}\int_\Omega |\nabla f_\delta|^p \, dx = \int_{\mathbb{M}^{2\times 2}} |M|^p \, d\nu_\delta(M).$$

# Iterating the Proposition

Let $C = \lambda A + (1-\lambda)B$ with $\lambda \in [0,1]$ and $\mathrm{rank}(B-A) = 1$. Let $f\colon \Omega \to \mathbb{R}^2$ such that $f(x) = Cx$ on $\partial\Omega$,

$$\mathrm{dist}(\nabla f, \{A, B\}) < \delta \quad \text{a.e. in} \quad \Omega.$$

**Further splitting:** $B = \mu B_1 + (1-\mu)B_2$ with $\mu \in [0,1]$, $\mathrm{rank}(B_2 - B_1) = 1$.

**New gradient:** apply previous Proposition to the set $\{x \in \Omega\colon \nabla f \sim B\}$ to obtain $\tilde{f}\colon \Omega \to \mathbb{R}^2$ such that $f(x) = Cx$ on $\partial\Omega$,

$$\mathrm{dist}(\nabla \tilde{f}, \{A, B_1, B_2\}) < \delta \quad \text{a.e. in} \quad \Omega.$$

The gradient distribution of $\tilde{f}$ is given by

$$\nu = \lambda\,\delta_A + (1-\lambda)\mu\,\delta_{B_1} + (1-\lambda)(1-\mu)\,\delta_{B_2}\,.$$

# Laminates of finite order

**Laminates of finite order:** laminates obtained iteratively through the splitting procedure in the previous slide.

## Proposition (Convex integration)

*Let* $\nu = \sum_{i=1}^{N} \lambda_i \delta_{A_i}$ *be a laminate of finite order, s.t.*

- $\overline{\nu} = A$,
- $A = \sum_{i=1}^{N} \lambda_i A_i$ *with* $\sum_{i=1}^{N} \lambda_i = 1$.

*Fix* $\delta > 0$. $\exists$ *a* **piecewise affine Lipschitz** *map* $f : \Omega \to \mathbb{R}^2$ *s.t.* $\nabla f \sim \nu$, *that is,*

- $\mathrm{dist}(\nabla f, \mathrm{supp}\, \nu) < \delta$ *a.e. in* $\Omega$,
- $f(x) = Ax$ *on* $\partial\Omega$,
- $|\{x \in \Omega : \ |\nabla f(x) - A_i| < \delta\}| = \lambda_i \, |\Omega|$.

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# Strategy of the Proof

**Strategy:** explicit construction of $u_n$ by **convex integration methods**.

1. Rewrite the equation $\operatorname{div}(\sigma \nabla u) = 0$ as a differential inclusion
$$\nabla f(x) \in T\,, \quad \text{for a.e.} \quad x \in \Omega \tag{2.3}$$
for $f \colon \Omega \to \mathbb{R}^2$ and an appropriate target set $T \subset \mathbb{M}^{2\times 2}$.
**Note:** $u$ and $f$ have the **same** integrability.
2. Construct a laminate $\nu$ with $\operatorname{supp} \nu \subset T$ and the right integrability.
3. Convex integration Proposition $\implies$ construct $f \colon \Omega \to \mathbb{R}^2$ s.t. $\nabla f \sim \nu$.
In this way $f$ solves (2.3) and
$$\nabla f \in L^q_{\text{weak}}(\Omega; \mathbb{R}^2)\,, \;\; q \in \left( \frac{2K}{K+1} - \delta, \frac{2K}{K+1} \right]\,, \qquad \nabla f \notin L^{\frac{2K}{K+1}}(\Omega; \mathbb{R}^2)\,.$$

These methods were developed for isotropic conductivities $\sigma \in L^\infty(\Omega; \{KI, \frac{1}{K} I\})$.
The adaptation to our case is non-trivial because of the lack of symmetry of the target set $T$, due to the anisotropy of $\sigma_1$ and $\sigma_2$.

Astala, Faraco, Székelyhidi. *Convex integration and the $L^p$ theory of elliptic equations.*
Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2008)

# Rewriting the PDE as a differential inclusion

Let $K > 1$, $S_1, S_2 \in [1/K, K]$ and define

$$\sigma_1 := \mathrm{diag}(1/K, 1/S_1)\,, \quad \sigma_2 := \mathrm{diag}(K, S_2)\,, \qquad \sigma := \sigma_1 \chi_{E_1} + \sigma_2 \chi_{E_2}\,,$$

$$T_1 := \left\{ \begin{pmatrix} x & -y \\ S_1^{-1}\, y & K^{-1}\, x \end{pmatrix} : \, x, y \in \mathbb{R} \right\}, \quad T_2 := \left\{ \begin{pmatrix} x & -y \\ S_2\, y & K\, x \end{pmatrix} : \, x, y \in \mathbb{R} \right\}.$$

## Lemma (F., Palombaro '17)

*A function $u \in W^{1,1}(\Omega)$ is solution to*

$$\mathrm{div}(\sigma \nabla u) = 0$$

*iff there exists $v \in W^{1,1}(\Omega)$ such that $f = (u, v) \colon \Omega \to \mathbb{R}^2$ satisfies*

$$\nabla f(x) \in T_1 \cup T_2 \quad \textit{in} \quad \Omega\,.$$

*Moreover $E_1 = \{x \in \Omega \colon \nabla f(x) \in T_1\}$ and $E_2 = \{x \in \Omega \colon \nabla f(x) \in T_2\}$.*

**Key Remark:** $u$ and $f$ enjoy the **same** integrability properties.

# Targets in conformal coordinates

**Conformal coordinates:** Let $A \in \mathbb{M}^{2\times 2}$. Then $A = (a_+, a_-)$ for $a_+, a_- \in \mathbb{C}$, defined by

$$Aw = a_+ w + a_- \overline{w}\,, \quad \forall\, w \in \mathbb{C}\,.$$

The sets of conformal linear maps and anti-conformal linear maps are

$$E_0 := \{(z,0) :\ z \in \mathbb{C}\} \qquad \text{(Conformal maps)}$$

$$E_\infty := \{(0,z) :\ z \in \mathbb{C}\} \qquad \text{(Anti-conformal maps)}$$

**Target sets** in conformal coordinates are

$$T_1 = \{(a, d_1(\overline{a})) \ :\ a \in \mathbb{C}\}\,, \qquad T_2 = \{(a, -d_2(\overline{a})) \ :\ a \in \mathbb{C}\}\,,$$

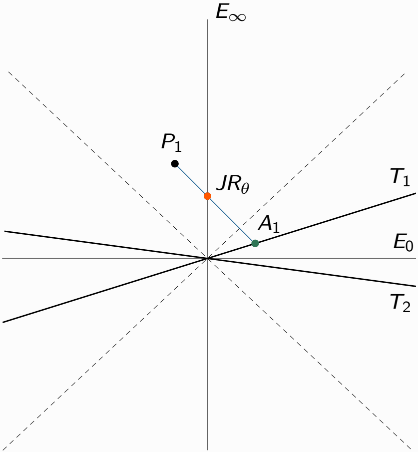
where the operators $d_j \colon \mathbb{C} \to \mathbb{C}$ are defined as

$$d_j(a) := k \operatorname{Re} a + i\, s_j \operatorname{Im} a\,, \quad \text{with} \quad k := \frac{K-1}{K+1} \quad \text{and} \quad s_j := \frac{S_j - 1}{S_j + 1}\,.$$

# Staircase Laminate (F., Palombaro '17)

Let $\theta \in [0, 2\pi]$, $JR_\theta = (0, e^{i\theta})$.

$JR_\theta = \lambda_1 A_1 + (1 - \lambda_1) P_1$



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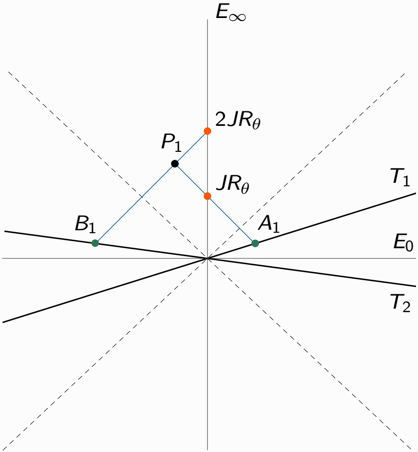
Let $\theta \in [0, 2\pi]$, $JR_\theta = (0, e^{i\theta})$.

$$JR_\theta = \lambda_1 A_1 + (1-\lambda_1) P_1$$
$$= \lambda_1 A_1 + (1-\lambda_1)(\mu_1 B_1 + (1-\mu_1)\, 2JR_\theta)$$
$$\rightsquigarrow \nu_1$$

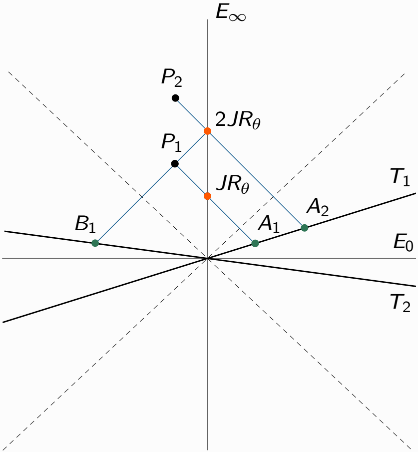


# Staircase Laminate (F., Palombaro '17)

Let $\theta \in [0, 2\pi]$, $JR_\theta = (0, e^{i\theta})$.

$$
\begin{aligned}
JR_\theta &= \lambda_1 A_1 + (1-\lambda_1) P_1 \\
&= \lambda_1 A_1 + (1-\lambda_1)(\mu_1 B_1 + (1-\mu_1)\, 2JR_\theta) \\
&\rightsquigarrow \nu_1
\end{aligned}
$$

$$
2JR_\theta = \lambda_2 A_2 + (1-\lambda_2) P_2
$$



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\begin{aligned}
JR_\theta &= \lambda_1 A_1 + (1-\lambda_1) P_1 \\
&= \lambda_1 A_1 + (1-\lambda_1)(\mu_1 B_1 + (1-\mu_1)\, 2JR_\theta) \\
&\rightsquigarrow \nu_1 \\
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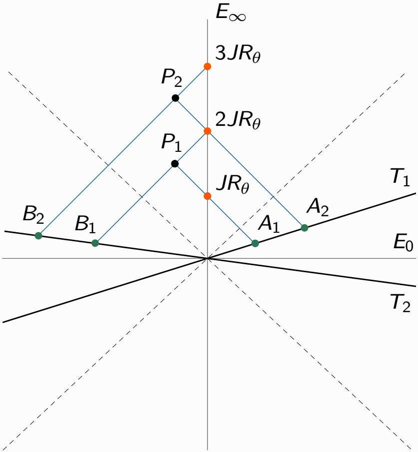
&= \lambda_2 A_2 + (1-\lambda_2)(\mu_2 B_2 + (1-\mu_2)\, 3JR_\theta) \\
&\rightsquigarrow \nu_2
\end{aligned}
$$



# Staircase Laminate (F., Palombaro '17)

Let $\theta \in [0, 2\pi]$, $JR_\theta = (0, e^{i\theta})$.

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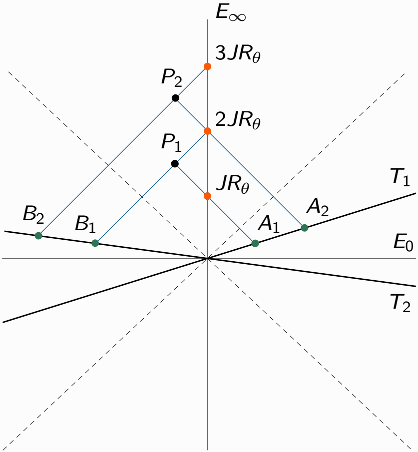
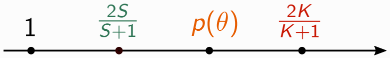
**Lemma:** $\exists\ p(\theta) \in \left[\frac{2S}{S+1}, \frac{2K}{K+1}\right]$ continuous, with $p(0) = \frac{2K}{K+1}$ and a sequence $\nu_n$ of laminates s.t.



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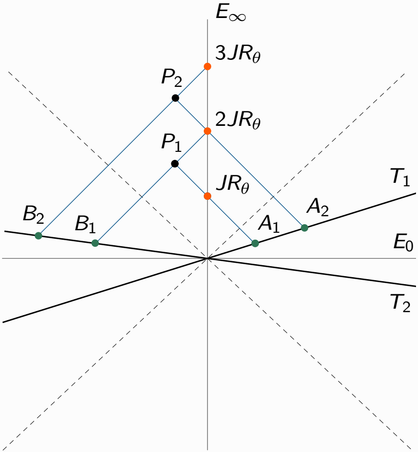


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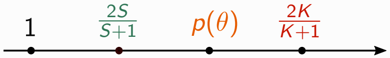
- $\operatorname{supp} \nu_n \subset T_1 \cup T_2 \cup E_\infty$
- $\overline{\nu}_n = JR_\theta$
- $\int_{\mathbb{M}^{2\times 2}} |M|^q \, d\nu_n(M) < \infty, \quad \forall\, q < p(\theta)$
- $\int_{\mathbb{M}^{2\times 2}} |M|^{p(\theta)} \, d\nu_n(M) \to \infty$ as $n \to \infty$



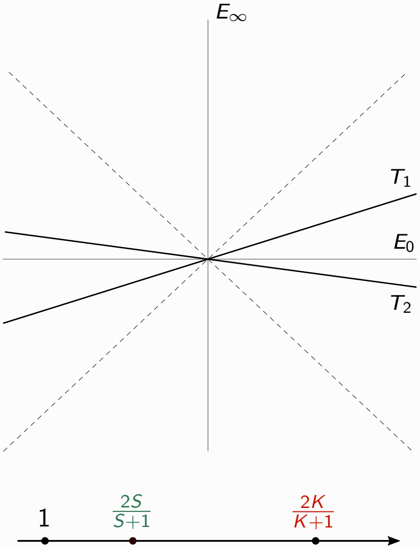

**Remark:** barycentre $J$ gives the right growth.

# Constructing approximate solutions

We want to construct $f\colon \Omega \to \mathbb{R}^2$ such that

- $\operatorname{dist}(\nabla f, T_1 \cup T_2) < \varepsilon$ a.e. in $\Omega$,
- $f = Jx$ on $\partial\Omega$,
- $\nabla f \in L^q_{\mathrm{weak}}$, $q \in I_\delta := \left(\frac{2K}{K+1} - \delta, \frac{2K}{K+1}\right]$,
- $\nabla f \notin L^{\frac{2K}{K+1}}$.

**Idea:** alternate one step of the staircase laminate with the convex integration Proposition.



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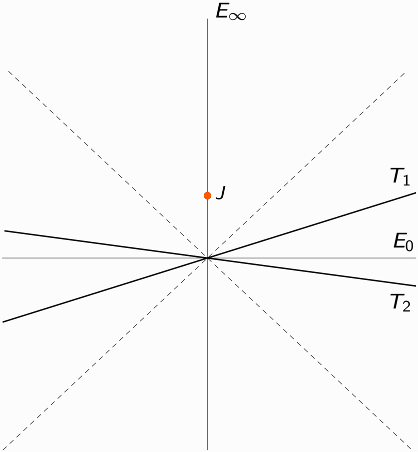


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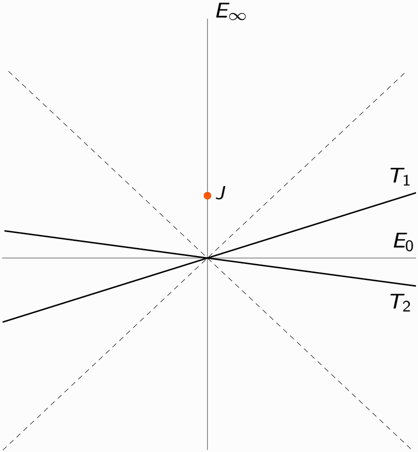


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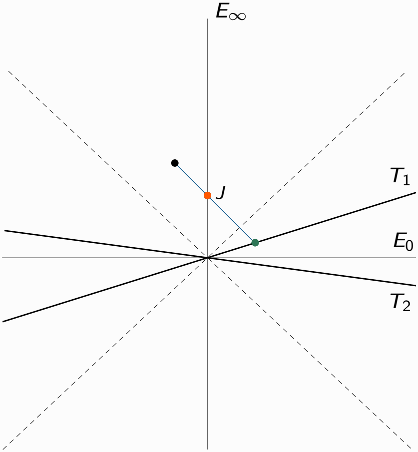


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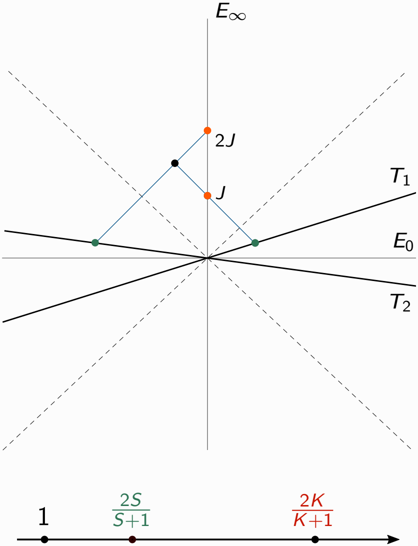

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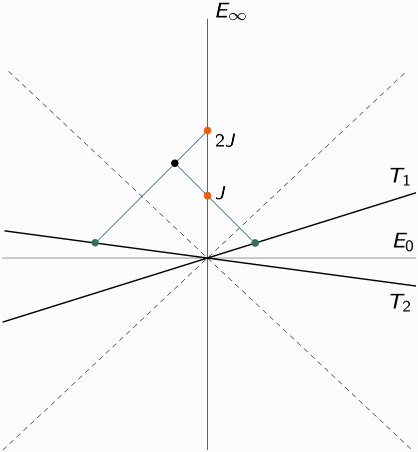


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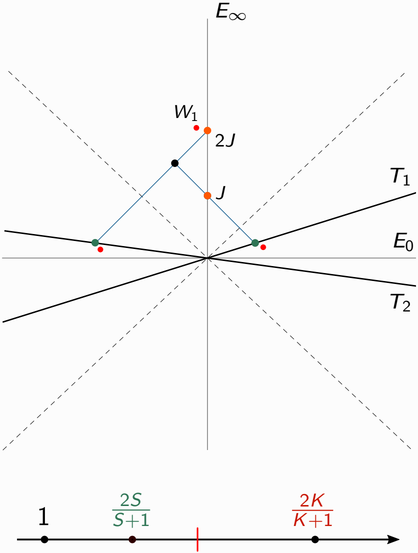

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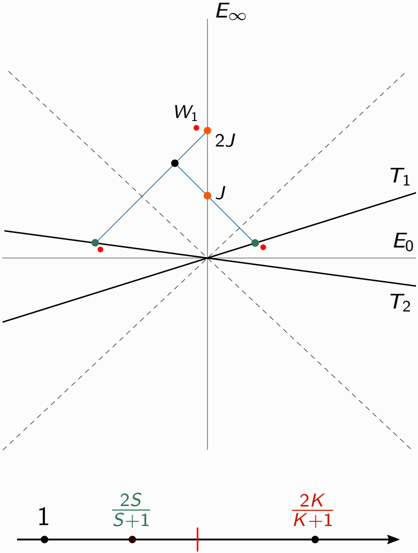

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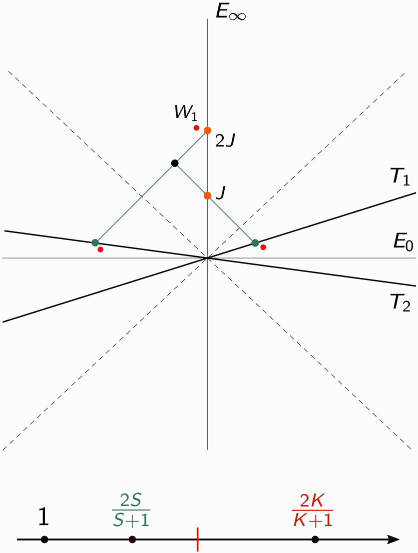

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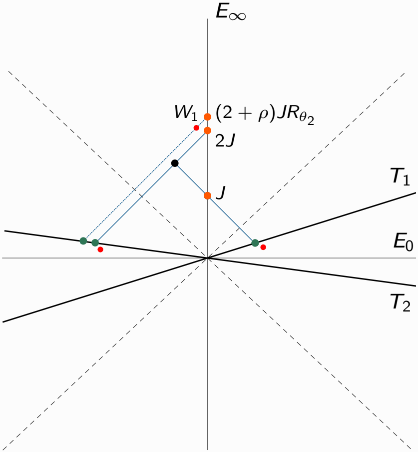


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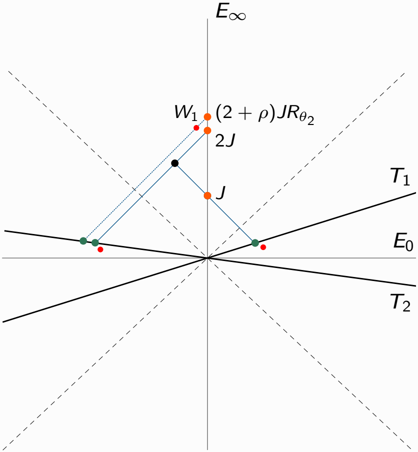


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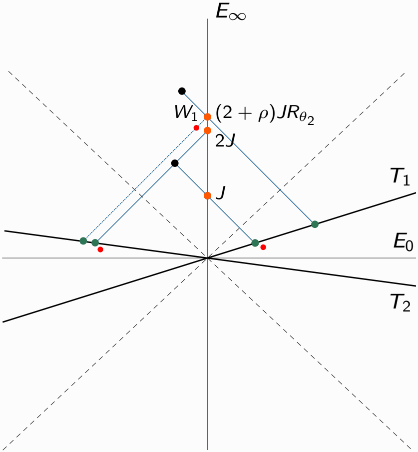


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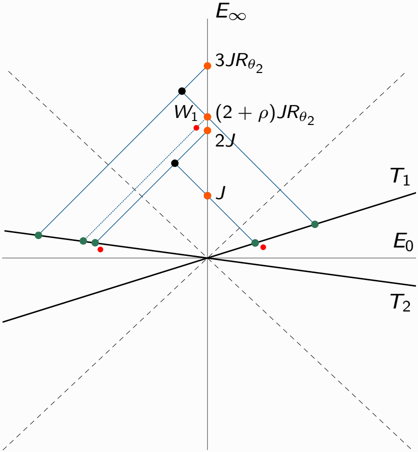


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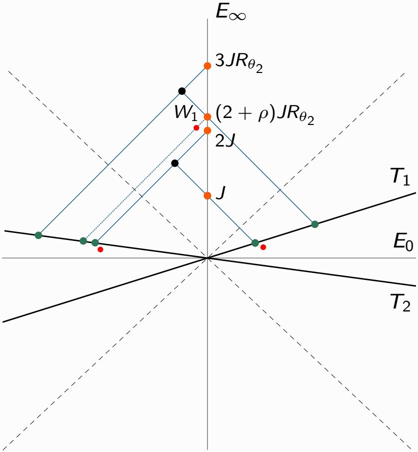


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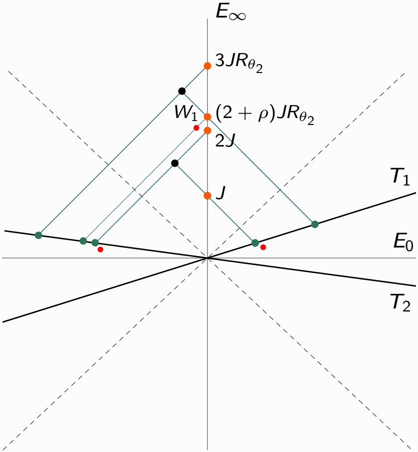


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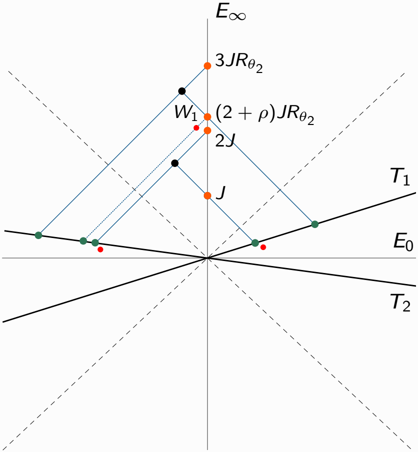


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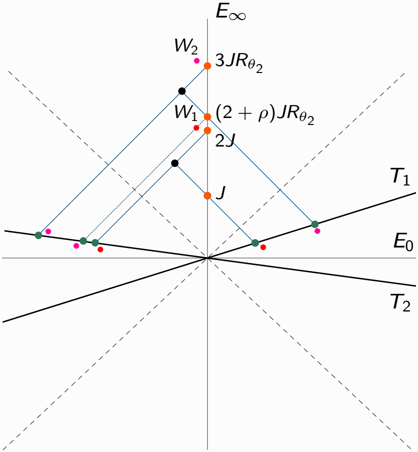


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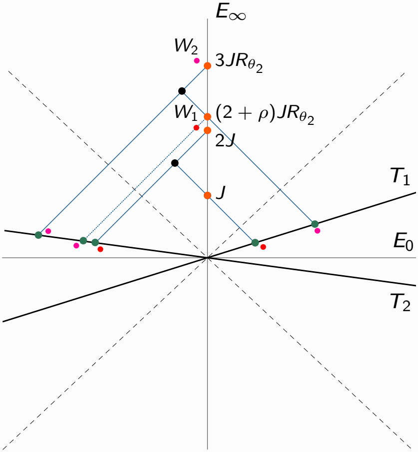
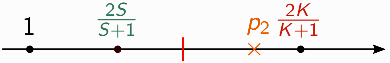
- Set $f_3 := g$ in the set $\{\nabla f_2 \sim W_1\}$ and $f_3 := f_2$ otherwise $\implies \nabla f_3$ grows like $p_2$



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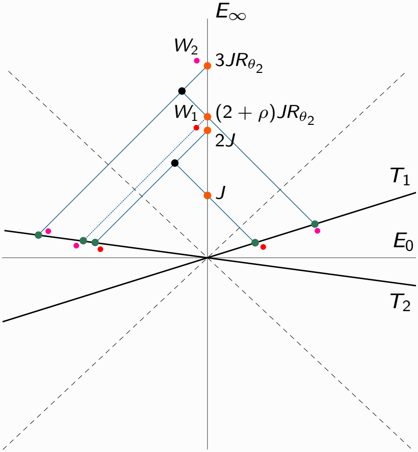


# Constructing approximate solutions

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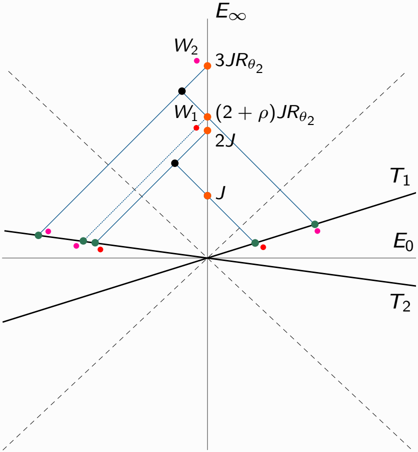


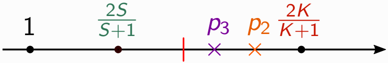

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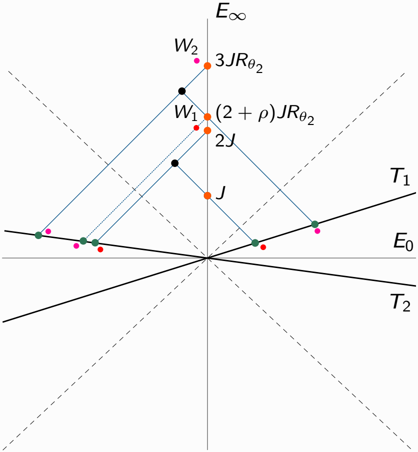
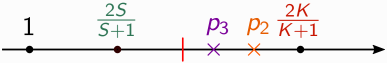
**Iterating:** $\rightsquigarrow f_n$ obtained by successive modifications on nested sets going to zero in measure $\implies f_n \to f$



# Conclusions and Perspectives

**Conclusions:** analysis of critical integrability of distributional solutions to

$$\operatorname{div}(\sigma\nabla u) = 0\,, \quad \text{in } \Omega\,, \tag{2.4}$$

when $\sigma \in \{\sigma_1, \sigma_2\}$ for $\sigma_1, \sigma_2 \in \mathbb{M}^{2\times 2}$ elliptic.

- Optimal exponents $q_{\sigma_1,\sigma_2}$ and $p_{\sigma_1,\sigma_2}$ were already characterised and the upper exponent $p_{\sigma_1,\sigma_2}$ was proved to be optimal.
  Nesi, Palombaro, Ponsiglione. Ann. Inst. H. Poincaré Anal. Non Linéaire (2014).
- We proved the optimality of the lower critical exponent $q_{\sigma_1,\sigma_2}$.

**Perspectives:**

- Stronger result for lower critical exponent: showing $\exists\ u \in W^{1,1}(\Omega)$ solution to (2.4) and s.t. $\nabla u \in L^{\frac{2K}{K+1}}_{\mathrm{weak}}(\Omega;\mathbb{R}^2)$ but $\nabla u \notin L^{\frac{2K}{K+1}}(B;\mathbb{R}^2)$, $\forall$ ball $B \subset \Omega$.
  Modifying staircase laminate?
- Extend these results to three-phase conductivities $\sigma \in \{\sigma_1, \sigma_2, \sigma_3\}$.
- Dimension $d \geq 3$? Even only in the isotropic case $\sigma \in \{KI, K^{-1}I\}$ for $K > 1$.
  Main difficulty: Astala's Theorem is missing in higher dimensions.


Thank You!