Optimal lower exponent of solutions to two-phase elliptic equations in two dimensions

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Problem

$\Omega \subset \mathbb{R}^2$ bounded open domain. A map $\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2})$ is **uniformly elliptic** if

$$\sigma\xi \cdot \xi \geq \lambda |\xi|^2 , \ \sigma^{-1}\xi \cdot \xi \geq \lambda |\xi|^2 \qquad \forall \xi \in \mathbb{R}^2, x \in \Omega .$$

Problem

Study the gradient integrability of distributional solutions $u \in W^{1,1}(\Omega)$ to

$$\operatorname{div}(\sigma \nabla u) = 0 , \tag{0.1}$$

when

$$\sigma = \sigma_1 \chi_{E_1} + \sigma_2 \chi_{E_2} ,$$

with $\sigma_1, \sigma_2 \in \mathbb{M}^{2\times 2}$ constant elliptic matrices, $\{E_1, E_2\}$ measurable partition of Ω.

Application to composites:

- Ω is a section of a composite conductor obtained by mixing two materials with conductivities σ_1 and σ_2
- the electric field ∇u solves (0.1)
- How much can ∇u concentrate, given the geometry $\{E_1, E_2\}$?

Astala's Theorem

Theorem (Astala '94)

Let $\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2})$ *be uniformly elliptic. There exists exponents* $1 < q < 2 < p$ *such that if* $u \in W^{1,q}(\Omega)$ *solves*

$$\operatorname{div}(\sigma \nabla u) = 0\,,$$

then $\nabla u \in L^p_{\text{weak}}(\Omega; \mathbb{R}^2)$.

Question

Are the exponents q *and* p *optimal among two-phase elliptic conductivities*

$$\sigma = \sigma_1 \chi_{E_1} + \sigma_2 \chi_{E_2}\,?$$

Astala. *Area distortion of quasiconformal mappings.* Acta Mathematica (1994)

Astala's exponents for two-phase conductivities

For two-phase conductivities Astala's exponents $q = q_{\sigma_1,\sigma_2}$ and $p = p_{\sigma_1,\sigma_2}$ have been characterised.

Remark: it is sufficient to prove optimality in the case

$$\sigma_1 = \begin{pmatrix} 1/K & 0 \\ 0 & 1/S_1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} K & 0 \\ 0 & S_2 \end{pmatrix},$$

where

$$K > 1 \qquad \text{and} \qquad \frac{1}{K} \leq S_j \leq K\,, \quad j = 1, 2\,.$$

The corresponding critical exponents for Astala's theorem are

$$q_{\sigma_1,\sigma_2} = \frac{2K}{K+1}, \quad p_{\sigma_1,\sigma_2} = \frac{2K}{K-1}.$$

Nesi, Palombaro, Ponsiglione. Ann. Inst. H. Poincaré Anal. Non Linéaire (2014).

Upper exponent optimality

Theorem (Nesi, Palombaro, Ponsiglione '14)

Let $\sigma_1 = \text{diag}(1/K, 1/S_1), \sigma_2 = \text{diag}(K, S_2)$ *with* $K > 1$ *and* $S_1, S_2 \in [1/K, K]$.

(i) *If* $\sigma \in L^\infty(\Omega; \{\sigma_1, \sigma_2\})$ *and* $u \in W^{1,\frac{2K}{K+1}}(\Omega)$ *solves*

$$\text{div}(\sigma \nabla u) = 0 \tag{0.2}$$

then $\nabla u \in L^{\frac{2K}{K-1}}_{\text{weak}}(\Omega; \mathbb{R}^2)$.

(ii) *There exists* $\bar{\sigma} \in L^\infty(\Omega; \{\sigma_1, \sigma_2\})$ *and a weak solution* $\bar{u} \in W^{1,2}(\Omega)$ *to* (0.2) *with* $\sigma = \bar{\sigma}$, *satisfying affine boundary conditions and such that* $\nabla \bar{u} \notin L^{\frac{2K}{K-1}}(\Omega; \mathbb{R}^2)$.

Question we address

Is the lower exponent $\frac{2K}{K+1}$ optimal?

Lower exponent optimality

Theorem (F., Palombaro '17)

Let $\sigma_1 = \mathrm{diag}(1/K, 1/S_1), \sigma_2 = \mathrm{diag}(K, S_2)$ *with* $K > 1$ *and* $S_1, S_2 \in [1/K, K]$*. There exist*

- *coefficients* $\sigma_n \in L^\infty(\Omega; \{\sigma_1; \sigma_2\})$,
- *exponents* $p_n \in \left[1, \frac{2K}{K+1}\right]$,
- *functions* $u_n \in W^{1,1}(\Omega)$ *such that* $u_n(x) = x_1$ *on* $\partial\Omega$,

such that

$$\mathrm{div}(\sigma_n \nabla u_n) = 0\,,$$

$$\nabla u_n \in L^{p_n}_{\mathrm{weak}}(\Omega; \mathbb{R}^2), \quad p_n \to \frac{2K}{K+1}, \quad \nabla u_n \notin L^{\frac{2K}{K+1}}(\Omega; \mathbb{R}^2)\,.$$

F., Palombaro. Calculus of Variations and Partial Differential Equations (2017)

Solving differential inclusions

Theorem (Approximate solutions for two phases)

Let $A, B \in \mathbb{M}^{2\times 2}$, $C := \lambda A + (1-\lambda)B$ *with* $\lambda \in [0,1]$, *and* $\delta > 0$. *Assume that*

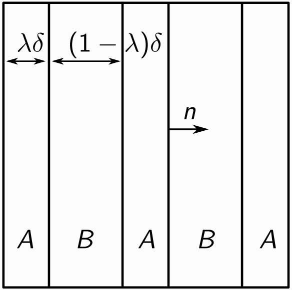
$$B - A = a \otimes n \quad \text{for some} \quad a \in \mathbb{R}^2, n \in S^1 \,. \qquad \text{(Rank-one connection)}$$

$\exists$ **piecewise affine Lipschitz** *map* $f : \Omega \to \mathbb{R}^2$ *such that* $f(x) = Cx$ *on* $\partial\Omega$ *and*

$$\mathrm{dist}(\nabla f, \{A, B\}) < \delta \quad \text{a.e. in} \quad \Omega \,.$$

Solutions: built through **simple laminates**

- rank-one connection allows to laminate in direction n,
- ∇f oscillates in δ-neighbourhoods of A and B,
- λ proportion for A, $1-\lambda$ proportion for B,
- this allows to recover boundary data C.



Müller. *Variational models for microstructure and phase transitions.*

Laminates of first order

$\mathcal{L}^2_\Omega$ is the normalised Lebesgue measure restricted to $\Omega \rightsquigarrow \mathcal{L}^2_\Omega(B) := |B \cap \Omega|/|\Omega|$.

Gradient distribution

Let $f \colon \Omega \to \mathbb{R}^2$ be Lipschitz. The **gradient distribution** of f is the Radon measure $\nabla f_\#(\mathcal{L}^2_\Omega)$ on $\mathbb{M}^{2\times 2}$ defined by

$$\nabla f_\#(\mathcal{L}^2_\Omega)(V) := \mathcal{L}^2_\Omega((\nabla f)^{-1}(V)), \quad \forall \text{ Borel set } V \subset \mathbb{M}^{2\times 2}.$$

Let f_δ be the map given by the previous Theorem. Then as $\delta \to 0$,

$$\nu_\delta := (\nabla f_\delta)_\#(\mathcal{L}^2_\Omega) \overset{*}{\rightharpoonup} \nu := \lambda \delta_A + (1-\lambda)\delta_B \quad \text{in} \quad \mathcal{M}(\mathbb{M}^{2\times 2}).$$

The measure ν is called a **laminate of first order**, and it encodes:

- **Oscillations** of ∇f_δ about $\{A, B\}$ and their proportions.
- **Boundary condition** since the barycentre of ν is $\overline{\nu} := \int_{\mathbb{M}^{2\times 2}} M \, d\nu(M) = C$.
- **Integrability** since for $p > 1$ we have

$$\frac{1}{|\Omega|} \int_\Omega |\nabla f_\delta|^p \, dx = \int_{\mathbb{M}^{2\times 2}} |M|^p \, d\nu_\delta(M).$$

Iterating the Proposition

Let $C = \lambda A + (1-\lambda)B$ with $\lambda \in [0,1]$ and $\text{rank}(B-A) = 1$. Let $f\colon \Omega \to \mathbb{R}^2$ such that $f(x) = Cx$ on $\partial\Omega$,

$$\text{dist}(\nabla f, \{A, B\}) < \delta \quad \text{a.e. in} \quad \Omega.$$

Further splitting: $B = \mu B_1 + (1-\mu)B_2$ with $\mu \in [0,1]$, $\text{rank}(B_2 - B_1) = 1$.

New gradient: apply previous Proposition to the set $\{x \in \Omega\colon \nabla f \sim B\}$ to obtain $\tilde{f}\colon \Omega \to \mathbb{R}^2$ such that $f(x) = Cx$ on $\partial\Omega$,

$$\text{dist}(\nabla \tilde{f}, \{A, B_1, B_2\}) < \delta \quad \text{a.e. in} \quad \Omega.$$

The gradient distribution of $\tilde{f}$ is given by

$$\nu = \lambda\, \delta_A + (1-\lambda)\mu\, \delta_{B_1} + (1-\lambda)(1-\mu)\, \delta_{B_2}\,.$$

Laminates of finite order

Laminates of finite order: laminates obtained iteratively through the splitting procedure in the previous slide.

Proposition (Convex integration)

Let $\nu = \sum_{i=1}^{N} \lambda_i \delta_{A_i}$ *be a laminate of finite order, s.t.*

- $\overline{\nu} = A$,
- $A = \sum_{i=1}^{N} \lambda_i A_i$ *with* $\sum_{i=1}^{N} \lambda_i = 1$.

Fix $\delta > 0$. $\exists$ *a* **piecewise affine Lipschitz** *map* $f : \Omega \to \mathbb{R}^2$ *s.t.* $\nabla f \sim \nu$, *that is,*

- $\operatorname{dist}(\nabla f, \operatorname{supp} \nu) < \delta$ *a.e. in* Ω,
- $f(x) = Ax$ *on* $\partial\Omega$,
- $|\{x \in \Omega : \ |\nabla f(x) - A_i| < \delta\}| = \lambda_i \, |\Omega|$.

Strategy of the Proof

Strategy: explicit construction of u_n by **convex integration methods**.

1. Rewrite the equation $\operatorname{div}(\sigma \nabla u) = 0$ as a differential inclusion
$$\nabla f(x) \in T\,, \qquad \text{for a.e.} \quad x \in \Omega \tag{0.3}$$
for $f\colon \Omega \to \mathbb{R}^2$ and an appropriate target set $T \subset \mathbb{M}^{2\times 2}$.
Note: u and f have the **same** integrability.
2. Construct a laminate ν with $\operatorname{supp} \nu \subset T$ and the right integrability.
3. Convex integration Proposition $\implies$ construct $f\colon \Omega \to \mathbb{R}^2$ s.t. $\nabla f \sim \nu$.
In this way f solves (0.3) and
$$\nabla f \in L^q_{\text{weak}}(\Omega;\mathbb{R}^2)\,,\ \ q \in \left(\frac{2K}{K+1} - \delta, \frac{2K}{K+1}\right]\,, \qquad \nabla f \notin L^{\frac{2K}{K+1}}(\Omega;\mathbb{R}^2)\,.$$

These methods were developed for isotropic conductivities $\sigma \in L^\infty(\Omega; \{KI, \frac{1}{K}I\})$.
The adaptation to our case is non-trivial because of the lack of symmetry of the target set T, due to the anisotropy of σ_1 and σ_2.

Astala, Faraco, Székelyhidi. *Convex integration and the L^p theory of elliptic equations.*
Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2008)

Rewriting the PDE as a differential inclusion

Let $K > 1$, $S_1, S_2 \in [1/K, K]$ and define

$$\sigma_1 := \mathrm{diag}(1/K, 1/S_1)\,, \quad \sigma_2 := \mathrm{diag}(K, S_2)\,, \qquad \sigma := \sigma_1 \chi_{E_1} + \sigma_2 \chi_{E_2}\,,$$

$$T_1 := \left\{ \begin{pmatrix} x & -y \\ S_1^{-1}\, y & K^{-1}\, x \end{pmatrix} : x, y \in \mathbb{R} \right\}, \quad T_2 := \left\{ \begin{pmatrix} x & -y \\ S_2\, y & K\, x \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

Lemma (F., Palombaro '17)

A function $u \in W^{1,1}(\Omega)$ is solution to

$$\mathrm{div}(\sigma \nabla u) = 0$$

iff there exists $v \in W^{1,1}(\Omega)$ such that $f = (u, v) \colon \Omega \to \mathbb{R}^2$ satisfies

$$\nabla f(x) \in T_1 \cup T_2 \quad \text{in} \quad \Omega\,.$$

Moreover $E_1 = \{x \in \Omega \colon \nabla f(x) \in T_1\}$ and $E_2 = \{x \in \Omega \colon \nabla f(x) \in T_2\}$.

Key Remark: u and f enjoy the **same** integrability properties.

Targets in conformal coordinates

Conformal coordinates: Let $A \in \mathbb{M}^{2\times 2}$. Then $A = (a_+, a_-)$ for $a_+, a_- \in \mathbb{C}$, defined by

$$Aw = a_+ w + a_- \overline{w}\,, \quad \forall\, w \in \mathbb{C}\,.$$

The sets of conformal linear maps and anti-conformal linear maps are

$$E_0 := \{(z,0) :\ z \in \mathbb{C}\} \qquad \text{(Conformal maps)}$$
$$E_\infty := \{(0,z) :\ z \in \mathbb{C}\} \qquad \text{(Anti-conformal maps)}$$

Target sets in conformal coordinates are

$$T_1 = \{(a, d_1(\overline{a})) \ :\ a \in \mathbb{C}\}\,, \qquad T_2 = \{(a, -d_2(\overline{a})) \ :\ a \in \mathbb{C}\}\,,$$

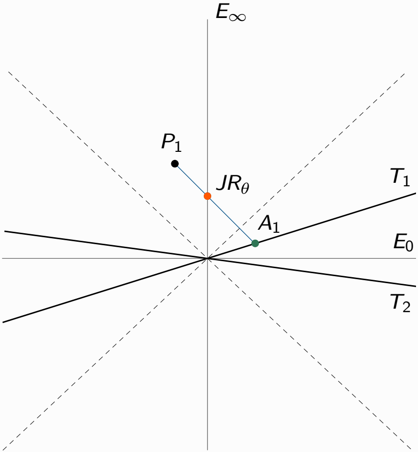
where the operators $d_j \colon \mathbb{C} \to \mathbb{C}$ are defined as

$$d_j(a) := k \operatorname{Re} a + i\, s_j \operatorname{Im} a\,, \quad \text{with} \quad k := \frac{K-1}{K+1} \quad \text{and} \quad s_j := \frac{S_j - 1}{S_j + 1}\,.$$

Staircase Laminate (F., Palombaro '17)

Let $\theta \in [0, 2\pi]$, $JR_\theta = (0, e^{i\theta})$.

$JR_\theta = \lambda_1 A_1 + (1 - \lambda_1) P_1$



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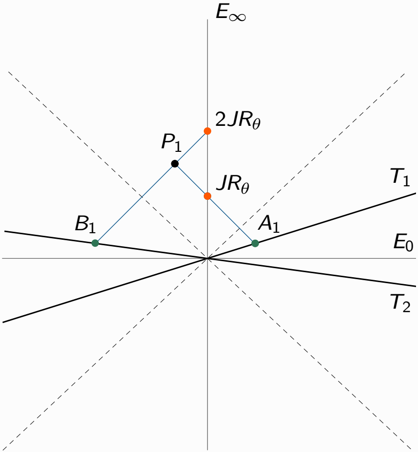

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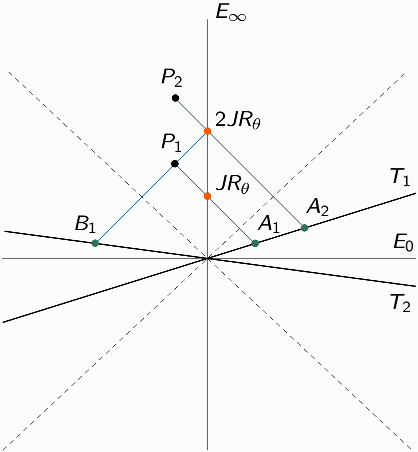

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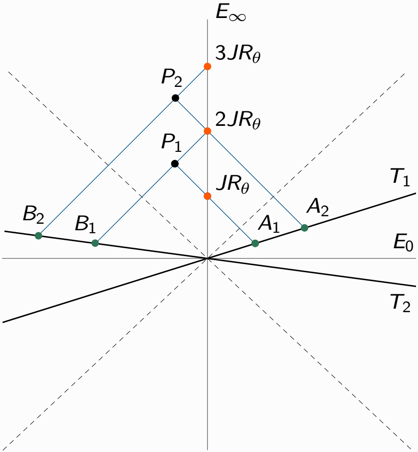
&= \lambda_2 A_2 + (1-\lambda_2)(\mu_2 B_2 + (1-\mu_2)\, 3JR_\theta) \\
&\rightsquigarrow \nu_2
\end{aligned}
$$



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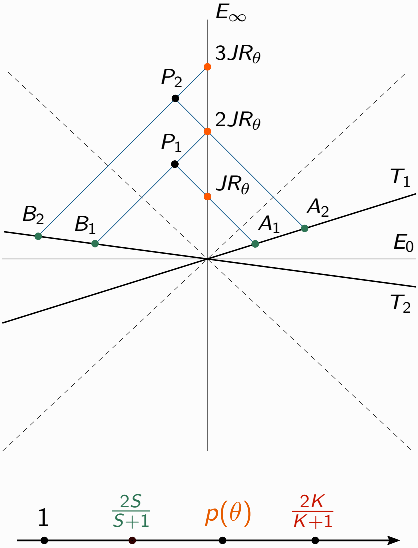
Lemma: $\exists\ p(\theta) \in \left[\frac{2S}{S+1}, \frac{2K}{K+1}\right]$ continuous, with $p(0) = \frac{2K}{K+1}$ and a sequence ν_n of laminates s.t.

- $\operatorname{supp} \nu_n \subset T_1 \cup T_2 \cup E_\infty$
- $\overline{\nu}_n = JR_\theta$
- $\int_{\mathbb{M}^{2\times 2}} |M|^q \, d\nu_n(M) < \infty, \quad \forall\, q < p(\theta)$
- $\int_{\mathbb{M}^{2\times 2}} |M|^{p(\theta)} \, d\nu_n(M) \to \infty$ as $n \to \infty$

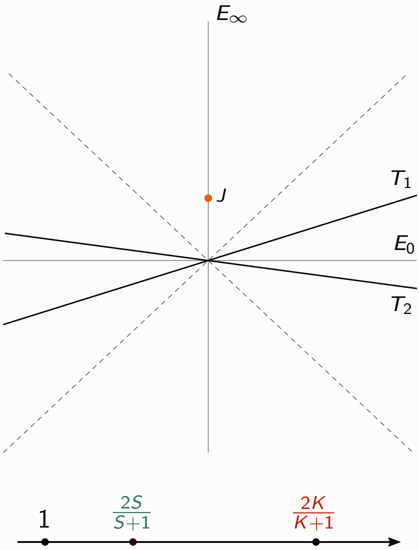


Remark: barycentre J gives the right growth.

Constructing approximate solutions

Recall $I_\delta := \left(\frac{2K}{K+1} - \delta, \frac{2K}{K+1}\right]$.

Step A. Define $f_1(x) := Jx \implies \theta_1 = 0, p_1 = \frac{2K}{K+1}$



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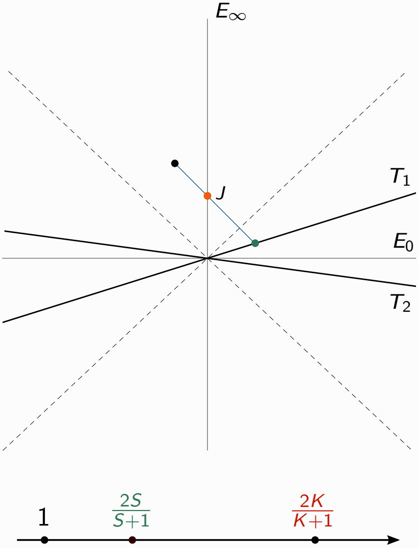

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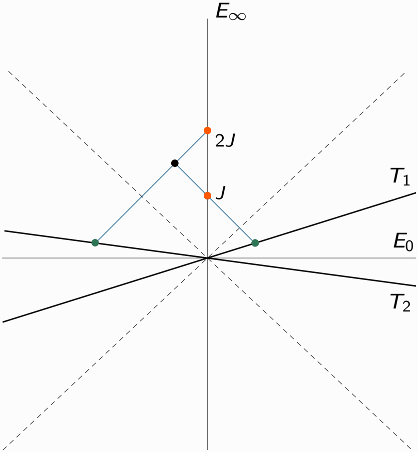
Step C. Proposition $\implies \exists$ map f_2 s.t. $f_2 = Jx$ on $\partial\Omega$
and $\nabla f_2 \sim \operatorname{supp} \nu_1 \implies \nabla f_2$ grows like p_1

This determines the exponent range I_δ



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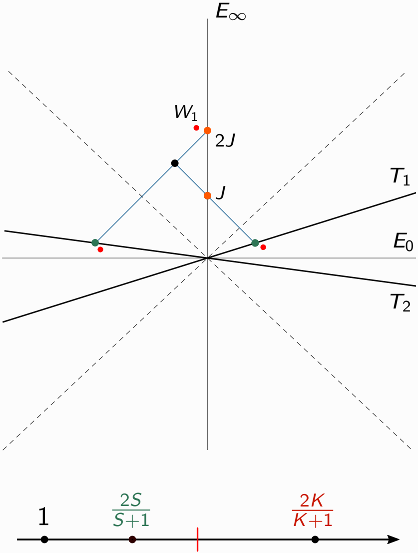

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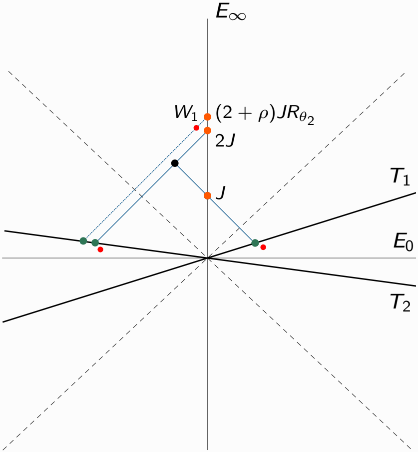
Step 1. One step of the staircase

- Split W_1. Since $W_1 \sim 2J \implies$ point $(2+\rho)JR_{\theta_2}$ with θ_2, ρ small. $\implies p_2 \in I_\delta$



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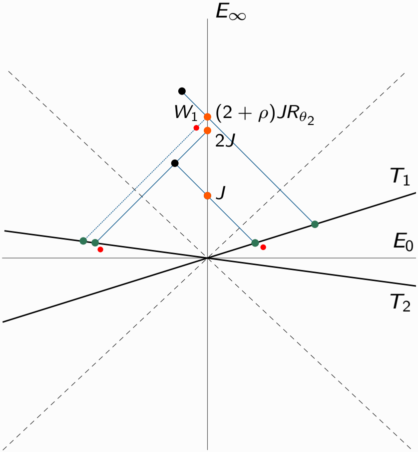

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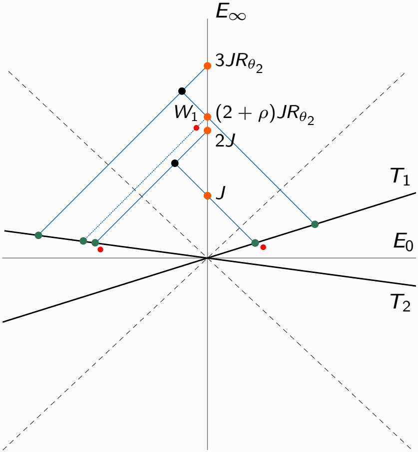

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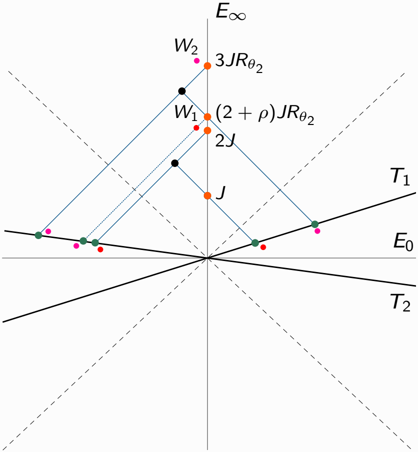
Step 2. Define map f_3 by modifying f_2

- Proposition $\implies \exists$ map g s.t. $g = W_1 x$ on $\partial\Omega$ and $\nabla g \sim \operatorname{supp} \nu_2 \implies \nabla g$ grows like p_2



Constructing approximate solutions

Recall $I_\delta := \left(\frac{2K}{K+1} - \delta, \frac{2K}{K+1}\right]$.

Step A. Define $f_1(x) := Jx \implies \theta_1 = 0, p_1 = \frac{2K}{K+1}$

Step B. Laminate ν_1 from J to $2J \rightsquigarrow$ growth p_1

Step C. Proposition $\implies \exists$ map f_2 s.t. $f_2 = Jx$ on $\partial\Omega$
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This determines the exponent range I_δ

Step 1. One step of the staircase

- Split W_1. Since $W_1 \sim 2J \implies$ point $(2+\rho)JR_{\theta_2}$ with θ_2, ρ small. $\implies p_2 \in I_\delta$
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Step 2. Define map f_3 by modifying f_2

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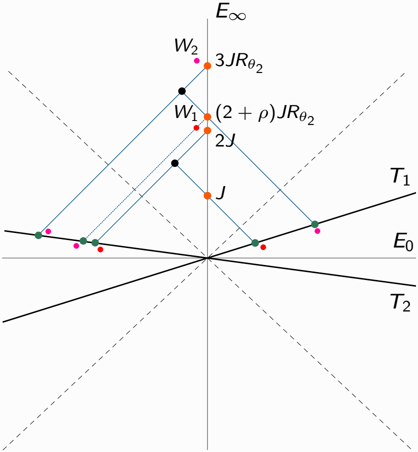

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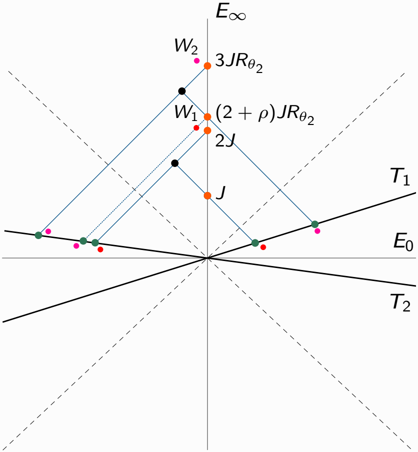

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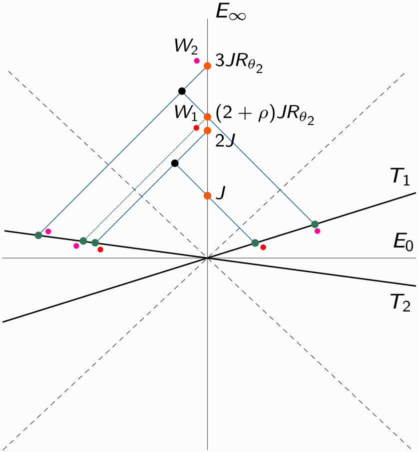

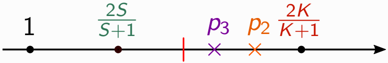

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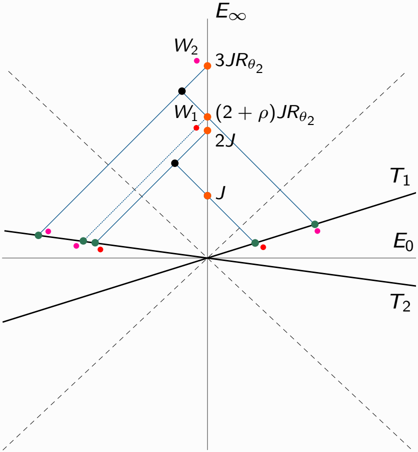
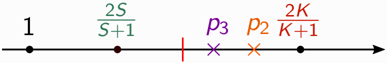
Iterating: $\rightsquigarrow f_n$ obtained by successive modifications on nested sets going to zero in measure $\implies f_n \to f$





Conclusions and Perspectives

Conclusions: analysis of critical integrability of distributional solutions to

$$\operatorname{div}(\sigma \nabla u) = 0\,, \quad \text{in} \;\; \Omega\,, \tag{0.4}$$

when $\sigma \in \{\sigma_1, \sigma_2\}$ for $\sigma_1, \sigma_2 \in \mathbb{M}^{2\times 2}$ elliptic.

- Optimal exponents q_{σ_1,σ_2} and p_{σ_1,σ_2} were already characterised and the upper exponent p_{σ_1,σ_2} was proved to be optimal.
  Nesi, Palombaro, Ponsiglione. Ann. Inst. H. Poincaré Anal. Non Linéaire (2014).
- We proved the optimality of the lower critical exponent q_{σ_1,σ_2}.

Perspectives:

- Stronger result for lower critical exponent: showing $\exists\, u \in W^{1,1}(\Omega)$ solution to (0.4) and s.t. $\nabla u \in L^{\frac{2K}{K+1}}_{\mathrm{weak}}(\Omega;\mathbb{R}^2)$ but $\nabla u \notin L^{\frac{2K}{K+1}}(B;\mathbb{R}^2)$, $\forall$ ball $B \subset \Omega$.
  Modifying staircase laminate?
- Extend these results to three-phase conductivities $\sigma \in \{\sigma_1, \sigma_2, \sigma_3\}$.
- Dimension $d \geq 3$? Even only in the isotropic case $\sigma \in \{KI, K^{-1}I\}$ for $K > 1$.
  Main difficulty: Astala's Theorem is missing in higher dimensions.

Thank You!