# Numbers Sequences and Series

**Revision Guide**

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# Revision Guide

Revision Guide for the Exam of the module **Numbers Sequences and Series 400297** 2024/25 at the University of Hull. If you have any question or find any typo, please email me at

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Full lenght Lecture Notes of the module available at

**silviofanzon.com/2024-NSS-Notes**

## Recommended revision strategy

Make sure you are very comfortable with:

1. The Definitions, Theorems, Proofs, and Examples contained in this Revision Guide
2. The Tutorial and Homework questions
3. The 2023/24 Exam Paper questions.
4. The Checklist below

## Checklist

You should be comfortable with the following topics/taks:

**Preliminaries**

- Prove that $\sqrt{p} \notin \mathbb{Q}$ for $p$ a prime number
- Compute infinite union / intersection
- Show that a binary relation is of equivalence / order / total order
- Characterize the equivalence classes of a given equivalence relation
- Prove statements by induction (such as Bernoulli's inequality)
- Compute the absolute value of a real number
- Understand how to apply triangle inequality

**Real Numbers**

- Determine if a given set with binary operation is a field
- Prove uniqueness of neutral element / inverse
- Computing Sup / Max and Inf / Min of a given set
- Prove that a given set is inductive
- Remember that $\mathbb{N}$, $\mathbb{Z}$ are not fields, $\mathbb{Q}$ is an ordered field, $\mathbb{R}$ is a complete ordered field
- State the axiom of completeness

**Properties of $\mathbb{R}$**

- Know how to use the Archimedean property
- Characterization of sup / inf in terms of $\varepsilon$
- Sup / Inf and Max / Min of intervals
- Determine if a given set is finite / countable / uncountable
- Remember that $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable
- Remember that $\mathbb{R}$ and the irrationals are uncountable

**Complex Numbers**

- Sum, multiplication, division, conjugate of complex numbers
- Computing the inverse of a complex number
- Find modulus and argument of a complex number
- Compute Cartesian, Trigonometric and Exponential form of a complex number
- Complex exponential and its properties
- Computing powers of complex numbers
- Solving degree 2 polynomial equations in $\mathbb{C}$
- Long division of polynomials
- Solving higher degree polynomial equations in $\mathbb{C}$
- Finding the roots of unity
- Finding the n-th roots of a complex number

**Sequences in $\mathbb{R}$**

- Use the definition of convergence to prove convergence of a given real sequence
- Prove that a given sequence is bounded
- Remember that convergent sequences are bounded
- Use the Algebra of Limits to prove convergence / divergence of a given sequence
- Use the Squeeze Theorem to prove convergence of a given sequence
- Use the Geometric Sequence Test to prove convergence / divergence of a given sequence
- Use the Ratio Test to prove convergence / divergence of a given sequence
- Prove that a sequence is monotone increasing / decreasing
- Know the statement of the Monotone Convergence Theorem
- Memorize the 4 Special Limits, and know how to apply them to study convergence / divergence of a given sequence

**Sequences in $\mathbb{C}$**

- Use the definition of convergence to prove convergence of a given complex sequence
- Prove that a complex sequence is bounded
- Use the Algebra of Limits to prove convergence / divergence of a given sequence
- Use the Geometric Sequence Test / Ratio Test to prove convergence / divergence of a given complex sequence
- Determine convergence of real and imaginary part of a given complex sequence

**Series**

- Compute the partial sums of a given series
- Compute the sum of a telescopic series
- Apply the Necessary Condition for Convergence to prove that a given series is divergent
- Use the Geometric Series test to determine convergence / divergence of a given geometric series
- Compute the sum of a given (convergent) geometric series
- Determine convergence / divergence of non-negative series by using the Cauchy Condensation Test, Comparison Test, Limit Comparison Test and Ratio Test
- Study convergence / divergence of $p$-series
- Prove that a given series converges absolutely
- Prove that a complex series converges / diverges by using the Ratio Test for general series
- Prove that a series converges conditionally
- Use the Dirichlet / Alternate Convergence / Abel's tests to study the convergence of a given series

# 1 Preliminaries

**Theorem 1.1**

The number $\sqrt{2}$ does not belong to $\mathbb{Q}$.

**Proof**

Aassume by contradiction that

$$\sqrt{2} \in \mathbb{Q}\,. \tag{1.1}$$

1. Therefore, there exist $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $n \neq 0$, such that
$$\frac{m}{n} = \sqrt{2}\,.$$
2. **Withouth loss of generality**, we can **assume** that $m$ and $n$ have no common factors.
3. Square the equation to get
$$\frac{m^2}{n^2} = 2 \quad \Longrightarrow \quad m^2 = 2n^2\,. \tag{1.2}$$
Therefore the integer $m^2$ is an even number.
4. Since $m^2$ is an even number, it follows that also $m$ is an even number. Then there exists $p \in \mathbb{N}$ such that
$$m = 2p\,. \tag{1.3}$$
5. Substitute (1.3) in (1.2) to get
$$m^2 = 2n^2 \quad \Longrightarrow \quad (2p)^2 = 2n^2 \quad \Longrightarrow \quad 4p^2 = 2n^2$$
Dividing both terms by 2, we obtain
$$n^2 = 2p^2\,. \tag{1.4}$$

Now, observe that:

- Equation (1.4) says that $n^2$ is even. The argument in Step 4 guarantees that also $n$ is even.
- Therefore $n$ and $m$ are both even, meaning they have 2 as common factor.
- But Step 2 says that $n$ and $m$ have no common factors. **Contradiction**

The **contradiction** stems from assumption (1.1). Thus, (1.1) is false, ending the proof.

## 1.1 Set Theory

**Definition 1.2**

Let $A$ be a set.

1. We write $x \in A$ if the element $x$ belongs to the set $A$.
2. We write $x \notin A$ if the element $x$ does not belong to the set $A$.

**Definition 1.3**

Given two sets $A$ and $B$, we say that $A$ is **contained** in $B$, in symbols

$$A \subseteq B\,,$$

if all the elements of $A$ are also contained in $B$. Two sets $A$ and $B$ are **equal**, in symbols

$$A = B\,,$$

if they contain the same elements.

**Remark 1.4**

The inclusion $A \subseteq B$ is equivalent to the implication:

$$x \in A \quad \Longrightarrow \quad x \in B$$

for all $x \in A$. The symbol $\Longrightarrow$ reads **implies**, and denotes the fact that the first condition implies the second.

**Definition 1.5:** Union and Intersection

For two sets $A$ and $B$ we define their **union** as the set

$$A \cup B := \{x \,:\, x \in A \text{ or } x \in B\}\,.$$

The **intersection** of $A$ and $B$ is defined by

$$A \cap B := \{x \,:\, x \in A \text{ and } x \in B\}\,.$$

We denote the **empty set** by the symbol $\emptyset$. Two sets are **disjoint** if

$$A \cap B = \emptyset\,.$$

**Proposition 1.6**

Let $A$ and $B$ be sets. Then

$$A = B \quad \Longleftrightarrow \quad A \subseteq B \text{ and } B \subseteq A\,.$$

**Definition 1.7:** Infinite union and intersection

Let $\Omega$ be a set, and $A_n \subseteq \Omega$ a family of subsets, where $n \in \mathbb{N}$.

1. The **infinte union** of the $A_n$ is the set
$$\bigcup_{n \in \mathbb{N}} A_n := \{x \in \Omega \,:\, x \in A_n \text{ for at least one } n \in \mathbb{N}\}\,.$$
2. The **infinte intersection** of the $A_n$ is the set
$$\bigcap_{n \in \mathbb{N}} A_n := \{x \in \Omega \,:\, x \in A_n \text{ for all } n \in \mathbb{N}\}\,.$$

**Example 1.8**

**Question.** Define $\Omega := \mathbb{N}$ and a family $A_n$ by

$$A_n = \{n, n+1, n+2, n+3, \dots\}, \quad n \in \mathbb{N}.$$

1. Prove that
$$\bigcup_{n\in\mathbb{N}} A_n = \mathbb{N}. \tag{1.5}$$

2. Prove that
$$\bigcap_{n\in\mathbb{N}} A_n = \emptyset. \tag{1.6}$$

**Solution.**

1. Assume that $m \in \cup_n A_n$. Then $m \in A_n$ for at least one $n \in \mathbb{N}$. Since $A_n \subseteq \mathbb{N}$, we conclude that $m \in \mathbb{N}$. This shows
$$\bigcup_{n\in\mathbb{N}} A_n \subseteq \mathbb{N}.$$
Conversely, suppose that $m \in \mathbb{N}$. By definition $m \in A_m$. Hence there exists at least one index $n$, $n = m$ in this case, such that $m \in A_n$. Then by definition $m \in \cup_{n\in\mathbb{N}} A_n$, showing that
$$\mathbb{N} \subseteq \bigcup_{n\in\mathbb{N}} A_n.$$
This proves (1.5).

2. Suppose that (1.6) is false, i.e.,
$$\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset.$$
This means there exists some $m \in \mathbb{N}$ such that $m \in \cap_{n\in\mathbb{N}} A_n$. Hence, by definition, $m \in A_n$ for all $n \in \mathbb{N}$. However $m \notin A_{m+1}$, yielding a contradiction. Thus (1.6) holds.

**Definition 1.9:** Complement

Let $A, B \subseteq \Omega$. The **complement** of $A$ with respect to $B$ is the set of elements of $B$ which do not belong to $A$, that is

$$B \setminus A := \{x \in \Omega : x \in B \text{ and } x \notin A\}.$$

In particular, the complement of $A$ with respect to $\Omega$ is denoted by

$$A^c := \Omega \setminus A := \{x \in \Omega : x \notin A\}.$$

**Example 1.10**

**Question.** Suppose $A, B \subseteq \Omega$. Prove that

$$A \subseteq B \iff B^c \subseteq A^c.$$

**Solution.** Let us prove the above claim:

- First implication $\implies$:
Suppose that $A \subseteq B$. We need to show that $B^c \subseteq A^c$. Hence, assume $x \in B^c$. By definition this means that $x \notin B$. Now notice that we cannot have that $x \in A$. Indeed, assume $x \in A$. By assumption we have $A \subseteq B$, hence $x \in B$. But we had assumed $x \in B$, contradiction. Therefore it must be that $x \notin A$. Thus $B^c \subseteq A^c$.

- Second implication $\impliedby$: Note that, for any set,
$$(A^c)^c = A.$$
Hence, by the first implication,
$$B^c \subseteq A^c \implies (A^c)^c \subseteq (B^c)^c \implies A \subseteq B.$$

**Proposition 1.11:** De Morgan's Laws

Suppose $A, B \subseteq \Omega$. Then

$$(A \cap B)^c = A^c \cup B^c, \qquad (A \cup B)^c = A^c \cap B^c.$$

**Definition 1.12**

Let $\Omega$ be a set. The **power set** of $\Omega$ is

$$\mathscr{P}(\Omega) := \{A : A \subseteq \Omega\}.$$

**Example 1.13**

**Question.** Compute the power set of

$$\Omega = \{x, y, z\}.$$

**Solution.** $\mathscr{P}(\Omega)$ has $2^3 = 8$, and

$$\begin{aligned}\mathscr{P}(\Omega) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}\\ \{x,z\}, \{y,z\}, \{x,y,z\}\}.\end{aligned}$$

**Definition 1.14:** Product of sets

Let $A, B$ be sets. The **product** of $A$ and $B$ is the set of pairs

$$A \times B := \{(a,b) : a \in A, b \in B\}.$$

## 1.2 Equivalence Relations

**Definition 1.15:** Binary relation

Suppose $A$ is a set. A **binary relation** $R$ on $A$ is a subset

$$R \subseteq A \times A.$$

**Definition 1.16:** Equivalence relation

A binary relation $R$ is called an **equivalence relation** if it satisfies the following properties:

1. **Reflexive**: For each $x \in A$ one has
$$(x, x) \in R,$$
2. **Symmetric**: We have
$$(x, y) \in R \implies (y, x) \in R$$
3. **Transitive**: We have
$$(x, y) \in R, \ (y, z) \in R \implies (x, z) \in R$$

If $(x, y) \in R$ we write

$$x \sim y$$

and we say that $x$ and $y$ are **equivalent**.

**Definition 1.17:** Equivalence classes

Suppose $R$ is an **equivalence relation** on $A$. The **equivalence class** of an element $x \in A$ is the set

$$[x] := \{y \in A \,:\, y \sim x\}.$$

The set of equivalence classes of elements of $A$ with respect to the equivalence relation $R$ is denoted by

$$A/R := A/\sim \,:= \{[x] \,:\, x \in A\}.$$

**Proposition 1.18:** Well-posedness of Definition 1.17

Let $\sim$ be an equivalence relation on $A$. Then

1. For each $x \in A$ we have $[x] \neq \emptyset$.
2. For all $x, y \in A$ it holds

$$x \sim y \quad \Longleftrightarrow \quad [x] = [y].$$

**Example 1.19:** Equality is an equivalence relation

**Question.** The equality defines a **binary relation** on $\mathbb{Q} \times \mathbb{Q}$, via

$$R := \{(x, y) \in \mathbb{Q} \times \mathbb{Q} \,:\, x = y\}.$$

1. Prove that $R$ is an **equivalence relation**.
2. Prove that $[x] = \{x\}$ and compute $\mathbb{Q}/R$.

**Solution.**

1. We need to check that $R$ satisfies the 3 properties of an equivalence relation:
   - Reflexive: It holds, since $x = x$ for all $x \in \mathbb{Q}$,
   - Symmetric: Again $x = y$ if and only if $y = x$,
   - Transitive: If $x = y$ and $y = z$ then $x = z$.

   Therefore, $R$ is an equivalence relation.
2. The class of equivalence of $x \in \mathbb{Q}$ is given by

$$[x] = \{x\},$$

   that is, this relation is quite trivial, given that each element of $\mathbb{Q}$ can only be related to itself. The quotient space is then

$$\mathbb{Q}/R = \{[x] \,:\, x \in \mathbb{Q}\} = \{\{x\} \,:\, x \in \mathbb{Q}\}.$$

**Example 1.20**

**Question.** Let $R$ be the binary relation on the set $\mathbb{Q}$ of rational numbers defined by

$$x \sim y \iff x - y \in \mathbb{Z}.$$

1. Prove that $R$ is an equivalence relation on $\mathbb{Q}$.
2. Compute $[x]$ for each $x \in \mathbb{Q}$.
3. Compute $\mathbb{Q}/R$.

**Solution.**

1. We have:
   - Reflexive: Let $x \in \mathbb{Q}$. Then $x - x = 0$ and $0 \in \mathbb{Z}$. Thus $x \sim x$.
   - Symmetric: If $x \sim y$ then $x - y \in \mathbb{Z}$. But then also

$$-(x - y) = y - x \in \mathbb{Z}$$

     and so $y \sim x$.
   - Transitive: Suppose $x \sim y$ and $y \sim z$. Then

$$x - y \in \mathbb{Z} \text{ and } y - z \in \mathbb{Z}.$$

     Thus, we have

$$x - z = (x - y) + (y - z) \in \mathbb{Z}$$

     showing that $x \sim z$.

   Thus, we have shown that $R$ is an equivalence relation on $\mathbb{Q}$.
2. Note that

$$x \sim y \quad \Longleftrightarrow \quad \exists\, n \in \mathbb{Z} \text{ s.t. } y = x + n.$$

   Therefore the equivalence classes with respect to $\sim$ are

$$[x] = \{x + n \,:\, n \in \mathbb{Z}\}.$$

   Each equivalence class has exactly one element in $[0, 1) \cap \mathbb{Q}$, meaning that:

$$\forall x \in \mathbb{Q}, \ \exists!\, q \in \mathbb{Q} \text{ s.t. } 0 \leq q < 1 \text{ and } q \in [x]. \tag{1.7}$$

   Indeed: take $x \in \mathbb{Q}$ arbitrary. Then $x \in [n, n+1)$ for some $n \in \mathbb{Z}$. Setting $q := x - n$ we obtain that

$$x = q + n, \qquad q \in [0, 1),$$

   proving (1.7). In particular (1.7) implies that for each $x \in \mathbb{Q}$ there exists $q \in [0, 1) \cap \mathbb{Q}$ such that

$$[x] = [q].$$

3. From Point 2 we conclude that

$$\mathbb{Q}/R = \{[x] \,:\, x \in \mathbb{Q}\} = \{q \in \mathbb{Q} \,:\, 0 \leq q < 1\}.$$

## 1.3 Order relations

**Definition 1.21:** Partial order

A binary relation $R$ on $A$ is called a **partial order** if it satisfies the following properties:

1. **Reflexive**: For each $x \in A$ one has

$$(x, x) \in R,$$

2. **Antisymmetric**: We have

$$(x, y) \in R \text{ and } (y, x) \in R \implies x = y$$

3. **Transitive**: We have

$$(x, y) \in R, \ (y, z) \in R \implies (x, z) \in R$$

**Definition 1.22:** Total order

A binary relation $R$ on $A$ is called a **total order relation** if it satisfies the following properties:

1. **Partial order**: $R$ is a partial order on $A$.
2. **Total**: For each $x, y \in A$ we have

$$(x, y) \in R \quad \text{or} \quad (y, x) \in R \,.$$

**Example 1.23:** Set inclusion is a partial order but not total order

**Question.** Let $\Omega$ be a non-empty set and consider its **power set**

$$\mathscr{P}(\Omega) = \{A \ : \ A \subseteq \Omega\} \,.$$

The inclusion defines **binary relation** on $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, via

$$R := \{(A, B) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \ : \ A \subseteq B\} \,.$$

1. Prove that $R$ is an **order relation**.
2. Prove that $R$ is **not a total order**.

**Solution.**

1. Check that $R$ is a partial order relation on $\mathscr{P}(\Omega)$:
   - Reflexive: It holds, since $A \subseteq A$ for all $A \in \mathscr{P}(\Omega)$.
   - Antisymmetric: If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
   - Transitive: If $A \subseteq B$ and $B \subseteq C$, then, by definition of inclusion, $A \subseteq C$.
2. In general, $R$ is **not** a total order. For example consider

$$\Omega = \{x, y\} \,.$$

   Thus

$$\mathscr{P}(\Omega) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\} \,.$$

   If we pick $A = \{x\}$ and $B = \{y\}$ then $A \cap B = \emptyset$, meaning that

$$A \not\subseteq B \,, \quad B \not\subseteq A \,.$$

   This shows $R$ is not a total order.

**Example 1.24:** Inequality is a total order

**Question.** Consider the binary relation

$$R := \{(x, y) \in \mathbb{Q} \times \mathbb{Q} \ : \ x \leq y\} \,.$$

Prove that $R$ is a **total order relation**.
**Solution.** We need to check that:

1. Reflexive: It holds, since $x \leq x$ for all $x \in \mathbb{Q}$,
2. Antisymmetric: If $x \leq y$ and $y \leq x$ then $x = y$.
3. Transitive: If $x \leq y$ and $y \leq z$ then $x \leq z$.

Finally, we halso have that $R$ is a **total order** on $\mathbb{Q}$, since for all $x, y \in \mathbb{Q}$ we have

$$x \leq y \quad \text{or} \quad y \leq x \,.$$

## 1.4 Induction

**Axiom 1.25:** Principle of Inducion

Let $\alpha(n)$ be a statement which depends on $n \in \mathbb{N}$. Suppose that

1. $\alpha(1)$ is true, and
2. Whenever $\alpha(n)$ is true, then $\alpha(n+1)$ is true.

Then $\alpha(n)$ is true for all $n \in \mathbb{N}$.

**Example 1.26:** Formula for summing first $n$ natural numbers

**Question.** Prove by induction that the following formula holds for all $n \in \mathbb{N}$:

$$1 + 2 + 3 + \ldots + (n-1) + n = \frac{n(n+1)}{2} \,. \tag{1.8}$$

**Solution**. Define

$$S(n) = 1 + 2 + \ldots + n \,.$$

This way the formula at (1.8) is equivalent to

$$S(n) = \frac{n(n+1)}{2} \,, \quad \forall \, n \in \mathbb{N} \,.$$

1. It is immediate to check that (1.8) holds for $n = 1$.
2. Suppose (1.8) holds for $n = k$. Then

$$\begin{aligned} S(k+1) &= 1 + \ldots + k + (k+1) \\ &= S(k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

   where in the first equality we used that (1.8) holds for $n = k$. We have proven that

$$S(k+1) = \frac{(k+1)(k+2)}{2} \,.$$

   The RHS in the above expression is exactly the RHS of (1.8) computed at $n = k+1$. Therefore, we have shown that formula (1.8) holds for $n = k + 1$.

By the Principle of Induction, we conclude that (1.8) holds for all $n \in \mathbb{N}$.

**Example 1.27:** Bernoulli's inequality

**Question.** Let $x \in \mathbb{R}$ with $x > -1$. Bernoulli's inequality states that

$$(1+x)^n \geq 1 + nx \,, \quad \forall\, n \in \mathbb{N}\,. \tag{1.9}$$

Prove Bernoulli's inequality by induction.
**Solution.** Let $x \in \mathbb{R}, x > -1$. We prove the statement by induction:

- Base case: (1.9) holds with equality when $n = 1$.
- Induction hypothesis: Let $k \in \mathbb{N}$ and suppose that (1.9) holds for $n = k$, i.e.,

  $$(1+x)^k \geq 1 + kx\,.$$

  Then

  $$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x)\\ &\geq (1+kx)(1+x)\\ &= 1 + kx + x + kx^2\\ &\geq 1 + (k+1)x\,,\end{aligned}$$

  where we used that $kx^2 \geq 0$. Then (1.9) holds for $n = k+1$.

By induction we conclude (1.9).

## 1.5 Absolute value

**Definition 1.28:** Absolute value

Let $x \in \mathbb{R}$. The **absolute value** of $x$ is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

**Proposition 1.29:** Properties of absolute value

For all $x \in \mathbb{R}$ they hold:

1. $|x| \geq 0$.
2. $|x| = 0$ if and only if $x = 0$.
3. $|x| = |-x|$.

**Lemma 1.30**

Let $x, y \in \mathbb{R}$. Then

$$|x| \leq y \iff -y \leq x \leq y\,.$$

**Corollary 1.31**

Let $x, y \in \mathbb{R}$. Then

$$|x| < y \iff -y < x < y\,.$$

**Theorem 1.32:** Triangle inequality

For every $x, y \in \mathbb{R}$ we have

$$||x| - |y|| \leq |x+y| \leq |x| + |y|\,. \tag{1.10}$$

**Proposition 1.33**

For any $x, y \in \mathbb{R}$ it holds

$$||x| - |y|| \leq |x-y| \leq |x| + |y|\,. \tag{1.11}$$

Moreover for any $x, y, z \in \mathbb{R}$ it holds

$$|x-y| \leq |x-z| + |z-y|\,.$$

# 2 Real Numbers

## 2.1 Fields

**Definition 2.1:** Binary operation

A binary operation on a set $K$ is a function

$$\circ \,:\, K \times K \to K$$

which maps the ordered pair $(x, y)$ into $x \circ y$.

**Definition 2.2:** Properties of binary operations

Let $K$ be a set and $\circ \,:\, K \times K \to K$ be a binary operation on $K$. We say that:

1. $\circ$ is **commutative** if

$$x \circ y = y \circ x\,, \quad \forall\, x, y \in K$$

2. $\circ$ is **associative** if

$$(x \circ y) \circ z = x \circ (y \circ z)\,, \quad \forall\, x, y, z \in K$$

3. An element $e \in K$ is called **neutral element** of $\circ$ if

$$x \circ e = e \circ x = x\,, \quad \forall\, x \in K$$

4. Let $e$ be a neutral element of $\circ$ and let $x \in K$. An element $y \in K$ is called an **inverse** of $x$ with respect to $\circ$ if

$$x \circ y = y \circ x = e\,.$$

**Example 2.3**

**Question.** Let $K = \{0, 1\}$ be a set with binary operation $\circ$ defined by the table

| $\circ$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 0 |

1. Is $\circ$ commutative? Justify your answer.
2. Is $\circ$ associative? Justify your answer.

**Solution.**

1. The operation $\circ$ is not commutative, since

$$0 \circ 1 = 1 \neq 0 = 1 \circ 0\,.$$

2. The operation $\circ$ is not associative, since

$$(0 \circ 1) \circ 1 = 1 \circ 1 = 0\,,$$

while

$$0 \circ (1 \circ 1) = 0 \circ 0 = 1\,,$$

so that

$$(0 \circ 1) \circ 1 \neq 0 \circ (1 \circ 1)\,.$$

## 2.2 Fields

**Definition 2.4:** Field

Let $K$ be a set with binary operations of **addition**

$$+ \,:\, K \times K \to K\,, \quad (x, y) \mapsto x + y$$

and **multiplication**

$$\cdot \,:\, K \times K \to K\,, \quad (x, y) \mapsto x \cdot y = xy\,.$$

We call the triple $(K, +, \cdot)$ a **field** if:

1. The addition $+$ satisfies: $\forall\, x, y, z \in K$
   - (A1) **Commutativity and Associativity**:

$$x + y = y + x$$

$$(x + y) + z = x + (y + z)$$

   - (A2) **Additive Identity**: There exists a **neutral element** in $K$ for $+$, which we call $0$. It holds:

$$x + 0 = 0 + x = x$$

   - (A3) **Additive Inverse**: There exists an **inverse** of $x$ with respect to $+$. We call this element the **additive inverse** of $x$ and denote it by $-x$. It holds

$$x + (-x) = (-x) + x = 0$$

2. The multiplication $\cdot$ satisifes: $\forall\, x, y, z \in K$
   - (M1) **Commutativity and Associativity**:

$$x \cdot y = y \cdot x$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

   - (M2) **Multiplicative Identity**: There exists a **neutral element** in $K$ for $\cdot$, which we call $1$. It holds:

$$x \cdot 1 = 1 \cdot x = x$$

   - (M3) **Multiplicative Inverse**: If $x \neq 0$ there exists an **inverse** of $x$ with respect to $\cdot$. We call this element the **multiplicative inverse** of $x$ and denote it by $x^{-1}$. It holds

$$x \cdot x^{-1} = x^{-1} \cdot x = 1$$

3. The operations $+$ and $\cdot$ are related by
   - (AM) **Distributive Property**: $\forall\, x, y, z \in K$

$$x \cdot (y + z) = (x \cdot y) + (y \cdot z)\,.$$

**Theorem 2.5**

Let $K$ with $+$ and $\cdot$ defined by

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Then $(K, +, \cdot)$ is a field.

**Definition 2.6:** Subtraction and division

Let $(K, +, \cdot)$ be a field. We define:

1. **Subtraction** as the operation $-$ defined by

$$x - y := x + (-y), \quad \forall\, x, y \in K,$$

where $-y$ is the additive inverse of $y$.

2. **Division** as the operation $/$ defined by

$$x/y := x \cdot y^{-1}, \quad \forall\, x, y \in K, \;\; y \neq 0,$$

where $y^{-1}$ is the multiplicative inverse of $y$.

**Proposition 2.7:** Uniqueness of neutral elements and inverses

Let $(K, +, \cdot)$ be a field. Then

1. There is a unique element in $K$ with the property of $0$.
2. There is a unique element in $K$ with the property of $1$.
3. For all $x \in K$ there is a unique additive inverse $-x$.
4. For all $x \in K$, $x \neq 0$, there is a unique multiplicative inverse $x^{-1}$.

**Proof**

1. Suppose that $0 \in K$ and $\tilde{0} \in K$ are both neutral element of $+$, that is, they both satisfy (A2). Then

$$0 + \tilde{0} = 0$$

since $\tilde{0}$ is a neutral element for $+$. Moreover

$$\tilde{0} + 0 = \tilde{0}$$

since $0$ is a neutral element for $+$. By commutativity of $+$, see property (A1), we have

$$0 = 0 + \tilde{0} = \tilde{0} + 0 = \tilde{0},$$

showing that $0 = \tilde{0}$. Hence the neutral element for $+$ is unique.
2. Exercise.
3. Let $x \in K$ and suppose that $y, \tilde{y} \in K$ are both additive inverses of $x$, that is, they both satisfy (A3). Therefore

$$x + y = 0$$

since $y$ is an additive inverse of $x$ and

$$x + \tilde{y} = 0$$

since $\tilde{y}$ is an additive inverse of $x$. Therefore we can use commutativity and associativity and of $+$, see property (A1), and the fact that $0$ is the neutral element of $+$, to infer

$$\begin{aligned} y &= y + 0 = y + (x + \tilde{y}) \\ &= (y + x) + \tilde{y} = (x + y) + \tilde{y} \\ &= 0 + \tilde{y} = \tilde{y}, \end{aligned}$$

concluding that $y = \tilde{y}$. Thus there is a unique additive inverse of $x$, and

$$y = \tilde{y} = -x,$$

with $-x$ the element from property (A3).
4. Exercise.

**Theorem 2.8**

Consider the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ with the usual operations $+$ and $\cdot$. We have:

- $(\mathbb{N}, +, \cdot)$ is **not a field**.
- $(\mathbb{Z}, +, \cdot)$ is **not a field**.
- $(\mathbb{Q}, +, \cdot)$ **is a field**.

## 2.3 Ordered fields

**Definition 2.9**

Let $K$ be a set with binary operations $+$ and $\cdot$, and with an order relation $\leq$. We call $(K, +, \cdot, \leq)$ an **ordered field** if:

1. $(K, +, \cdot)$ is a field
2. There $\leq$ is of **total order** on $K$: $\forall\, x, y, z \in K$
   - (O1) **Reflexivity**:
   $$x \leq x$$
   - (O2) **Antisymmetry**:
   $$x \leq y \text{ and } y \leq x \implies x = y$$
   - (O3) **Transitivity**:
   $$x \leq y \text{ and } y \leq z \implies x = z$$
   - (O4) **Total order**:
   $$x \leq y \text{ or } y \leq x$$
3. The operations $+$ and $\cdot$, and the total order $\leq$, are related by the following properties: $\forall x, y, z \in K$
   - (AM) **Distributive**: Relates addition and multiplication via
   $$x \cdot (y + z) = x \cdot y + x \cdot z$$
   - (AO) Relates addition and order with the requirement:
   $$x \leq y \implies x + z \leq y + z$$
   - (MO) Relates multiplication and order with the requirement:
   $$x \geq 0,\; y \geq 0 \implies x \cdot y \geq 0$$

**Theorem 2.10**

$(\mathbb{Q}, +, \cdot, \leq)$ is an **ordered field**.

## 2.4 Supremum and infimum

In the following we assume that $(K, +, \cdot, \leq)$ is an ordered field.

**Definition 2.11:** Upper bound and bounded above

Let $A \subseteq K$:

1. We say that $b \in K$ is an **upper bound** for $A$ if

$$a \leq b, \quad \forall a \in A.$$

2. We say that $A$ is **bounded above** if there exists and upper bound $b \in K$ for $A$.

**Definition 2.12:** Supremum

Let $A \subseteq K$. A number $s \in K$ is called **least upper bound** or **supremum** of $A$ if:

1. $s$ is an upper bound for $A$,
2. $s$ is the smallest upper bound of $A$, that is,

$$\text{If } b \in K \text{ is upper bound for } A \text{ then } s \leq b.$$

If it exists, the supremum is denoted by

$$s := \sup A.$$

**Remark 2.13**

Note that if a set $A \subseteq K$ in **NOT** bounded above, then the supremum does not exist, as there are no upper bounds of $A$.

**Proposition 2.14:** Uniqueness of the supremum

Let $A \subseteq K$. If $\sup A$ exists, then it is unique.

**Definition 2.15:** Maximum

Let $A \subseteq K$. A number $M \in K$ is called the **maximum** of $A$ if:

$$M \in A \text{ and } a \leq M, \forall a \in A.$$

If it exists, we denote the maximum by

$$M = \max A.$$

**Proposition 2.16:** Relationship between Max and Sup

Let $A \subseteq K$. If the maximum of $A$ exists, then also the supremum exists, and

$$\sup A = \max A.$$

**Definition 2.17:** Lower bound, bounded below, infimum, minimum

Let $A \subseteq K$:

1. We say that $l \in K$ is a **lower bound** for $A$ if

$$l \leq a, \quad \forall a \in A.$$

2. We say that $A$ is **bounded below** if there exists a lower bound $l \in K$ for $A$.
3. We say that $i \in K$ is the **greatest lower bound** or **infimum** of $A$ if:
   - $i$ is a lower bound for $A$,
   - $i$ is the largest lower bound of $A$, that is,

$$\text{If } l \in K \text{ is a lower bound for } A \text{ then } l \leq i.$$

   If it exists, the infimum is denoted by

$$i = \inf A.$$

4. We say that $m \in K$ is the **minimum** of $A$ if:

$$m \in A \text{ and } m \leq a, \forall a \in A.$$

   If it exists, we denote the minimum by

$$m = \min A.$$

**Proposition 2.18**

Let $A \subseteq K$:

1. If $\inf A$ exists, then it is unique.
2. If the minimum of $A$ exists, then also the infimum exists, and

$$\inf A = \min A.$$

**Proposition 2.19**

Let $A \subseteq K$. If $\inf A$ and $\sup A$ exist, then

$$\inf A \leq a \leq \sup A, \quad \forall a \in A.$$

**Proposition 2.20:** Relationship between sup and inf

Let $A \subseteq K$. Define

$$-A := \{-a : a \in A\}.$$

They hold

1. If $\sup A$ exists, then $\inf A$ exists and

$$\inf(-A) = -\sup A.$$

2. If $\inf A$ exists, then $\sup A$ exists and

$$\sup(-A) = -\inf A.$$

## 2.5 Axioms of Real Numbers

**Definition 2.21:** Completeness

Let $(K, +, \cdot, \leq)$ be an ordered field. We say that $K$ is **complete** if the following property holds:

- (AC) For every $A \subseteq K$ non-empty and bounded above

$$\sup A \in K\,.$$

**Theorem 2.22**

$\mathbb{Q}$ is not complete. In particular, there exists a set $A \subseteq \mathbb{Q}$ such that

- $A$ is non-empty,
- $A$ is bounded above,
- $\sup A$ does not exist in $\mathbb{Q}$.

One of such sets is, for example,

$$A = \{q \in \mathbb{Q} \,:\; q \geq 0\,,\; q^2 < 2\}\,.$$

**Proposition 2.23**

Let $(K, +, \cdot, \leq)$ be a complete ordered field. Suppose that $A \subseteq K$ is non-empty and bounded below. Then

$$\inf A \in K\,.$$

**Definition 2.24:** System of Real Numbers $\mathbb{R}$

A system of Real Numbers is a set $\mathbb{R}$ with two operations $+$ and $\cdot$, and a total order relation $\leq$, such that

- $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field
- $\mathbb{R}$ sastisfies the Axiom of Completeness

## 2.6 Inductive sets

**Definition 2.25:** Inductive set

Let $S \subseteq \mathbb{R}$. We say that $S$ is an inductive set if they are satisfied:

- $1 \in S$,
- If $x \in S$, then $(x+1) \in S$.

**Example 2.26**

**Question.** Prove the following:

1. $\mathbb{R}$ is an inductive set.
2. The set $A = \{0, 1\}$ is not an inductive set.

**Solution.**

1. We have that $1 \in \mathbb{R}$ by axiom (M2). Moreover $(x+1) \in \mathbb{R}$ for every $x \in \mathbb{R}$, by definition of sum $+$.
2. We have $1 \in A$, but $(1+1) \notin A$, since $1 + 1 \neq 0$.

**Proposition 2.27**

Let $\mathscr{M}$ be a collection of inductive subsets of $\mathbb{R}$. Then

$$S := \bigcap_{M \in \mathscr{M}} M$$

is an inductive subset of $\mathbb{R}$.

**Definition 2.28:** Set of Natural Numbers

Let $\mathscr{M}$ be the collection of **all** inductive subsets of $\mathbb{R}$. We define the set of natural numbers in $\mathbb{R}$ as

$$\mathbb{N} := \bigcap_{M \in \mathscr{M}} M\,.$$

**Proposition 2.29:** $\mathbb{N}_{\mathbb{R}}$ is the smallest inductive subset of $\mathbb{R}$

Let $C \subseteq \mathbb{R}$ be an inductive subset. Then

$$\mathbb{N} \subseteq C\,.$$

In other words, $\mathbb{N}$ is the smallest inductive set in $\mathbb{R}$.

**Theorem 2.30**

Let $x \in \mathbb{N}$. Then

$$x \geq 1\,.$$

# 3 Properties of $\mathbb{R}$

**Theorem 3.1:** Archimedean Property

Let $x \in \mathbb{R}$ be given. Then:

1. There exists $n \in \mathbb{N}$ such that
$$n > x .$$
2. Suppose in addition that $x > 0$. There exists $n \in \mathbb{N}$ such that
$$\frac{1}{n} < x .$$

**Theorem 3.2:** Archimedean Property (Alternative formulation)

Let $x, y \in \mathbb{R}$, with $0 < x < y$. There exists $n \in \mathbb{N}$ such that
$$nx > y .$$

**Theorem 3.3:** Nested Interval Property

For each $n \in \mathbb{N}$ assume given a closed interval
$$I_n := [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\} .$$
Suppose that the intervals are nested, that is,
$$I_n \supset I_{n+1} , \quad \forall n \in \mathbb{N} .$$
Then
$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset . \tag{3.1}$$

**Example 3.4**

**Question.** Consider the **open** intervals
$$I_n := \left(0, \frac{1}{n}\right) .$$
These are clearly nested
$$I_n \supset I_{n+1} , \quad \forall n \in \mathbb{N} .$$
Prove that
$$\bigcap_{n=1}^{\infty} I_n = \emptyset . \tag{3.2}$$
**Solution.** Suppose by contradiction that the intersection is non-empty. Then there exists $x \in \mathbb{N}$ such that
$$x \in I_n , \quad \forall n \in \mathbb{N} .$$
By definition of $I_n$ the above reads
$$0 < x < \frac{1}{n} , \quad \forall n \in \mathbb{N} . \tag{3.3}$$
Since $x > 0$, by the Archimedean Property in Theorem 3.1 Point 2, there exists $n_0 \in \mathbb{N}$ such that
$$0 < \frac{1}{n_0} < x .$$
The above contradicts (3.3). Therefore (3.2) holds.

## 3.1 Revisiting Sup and Inf

**Proposition 3.5:** Characterization of Supremum

Let $A \subseteq \mathbb{R}$ be a non-empty set. Suppose that $s \in \mathbb{R}$ is an upper bound for $A$. They are equivalent:

1. $s = \sup A$
2. For every $\varepsilon > 0$ there exists $x \in A$ such that
$$s - \varepsilon < x .$$

**Proposition 3.6:** Characterization of Infimum

Let $A \subseteq \mathbb{R}$ be a non-empty set. Suppose that $i \in \mathbb{R}$ is a lower bound for $A$. They are equivalent:

1. $i = \inf A$
2. For every $\varepsilon \in \mathbb{R}$, with $\varepsilon > 0$, there exists $x \in A$ such that
$$x < i + \varepsilon .$$

**Proposition 3.7**

Let $a, b \in \mathbb{R}$ with $a < b$. Let
$$A := (a, b) = \{x \in \mathbb{R} : a < x < b\} .$$
1. We have that
$$\inf A = a , \quad \sup A = b .$$
2. $\min A$ and $\max A$ do not exist.

**Corollary 3.8**

Let $a, b \in \mathbb{R}$ with $a < b$. Let
$$A := [a, b) = \{x \in \mathbb{R} : a \leq x < b\} .$$
Then
$$\min A = \inf A = a , \quad \sup A = b ,$$
$\max A$ does not exist.

**Proposition 3.9**

Define the set
$$A := \left\{ \frac{1}{n} \,:\, n \in \mathbb{N} \right\}.$$
Then
$$\inf A = 0\,, \quad \sup A = \max A = 1\,.$$

**Proof**

*Part 1.* We have
$$\frac{1}{n} \le 1\,, \quad \forall\, n \in \mathbb{N}\,.$$
Therefore 1 is an upper bound for $A$. Since $1 \in A$, by definition of maximum we conclude that
$$\max A = 1\,.$$
Since the maximum exists, we conclude that also the supremum exists, and
$$\sup A = \max A = 1\,.$$
*Part 2.* We have
$$\frac{1}{n} > 0\,, \quad \forall\, n \in \mathbb{N}\,,$$
showing that 0 is a lower bound for $A$. Suppose by contradiction that 0 is not the infimum. Therefore 0 is not the largest lower bound. Then there exists $\varepsilon \in \mathbb{R}$ such that:

- $\varepsilon$ is a lower bound for $A$, that is,
$$\varepsilon \le \frac{1}{n}\,, \quad \forall\, n \in \mathbb{N}\,, \tag{3.4}$$
- $\varepsilon$ is larger than 0:
$$0 < \varepsilon\,.$$

As $\varepsilon > 0$, by the Archimedean Property there exists $n_0 \in \mathbb{N}$ such that
$$0 < \frac{1}{n_0} < \varepsilon\,.$$
This contradicts (3.4). Thus 0 is the largest lower bound of $A$, that is, $0 = \inf A$.
*Part 3.* We have that $\min A$ does not exist. Indeed suppose by contradiction that $\min A$ exists. Then
$$\min A = \inf A\,.$$
As $\inf A = 0$ by Part 2, we conclude $\min A = 0$. As $\min A \in A$, we obtain $0 \in A$, which is a contradiction.

## 3.2 Cardinality

**Definition 3.10:** Bijective function

Let $X, Y$ be sets and $f : X \to Y$ be a function. We say that:

1. $f$ is **injective** if it holds:
$$f(x) = f(y) \quad \Longrightarrow \quad x = y\,.$$
2. $f$ is **surjective** if it holds:
$$\forall\, y \in Y\,,\ \exists\, x \in X \ \text{ s.t. } \ f(x) = y\,.$$
3. $f$ is **bijective** if it is both **injective** and **surjective**.

**Definition 3.11:** Cardinality, Finite, Countable, Uncountable

Let $X$ be a set. The **cardinality** of $X$ is the number of elements in $X$. We denote the cardinality of $X$ by
$$|X| := \# \text{ of elements in } X\,.$$
Further, we say that:

1. $X$ is **finite** if there exists a natural number $n \in \mathbb{N}$ and a bijection
$$f : \{1, 2, \dots, n\} \to X\,.$$
In particular
$$|X| = n\,.$$
2. $X$ is **countable** if there exists a bijection
$$f : \mathbb{N} \to X\,.$$
In this case we denote the cardinality of $X$ by
$$|X| = |\mathbb{N}|\,.$$
3. $X$ is **uncountable** if $X$ is neither finite, nor countable.

**Proposition 3.12**

Let $X$ be a countable set and $A \subseteq X$. Then either $A$ is finite or countable.

**Example 3.13**

**Question.** Prove that $X = \{a, b, c\}$ is finite.
**Solution.** Set $Y = \{1, 2, 3\}$. The function $f : X \to Y$ defined by
$$f(1) = a\,, \quad f(2) = b\,, \quad f(3) = c\,,$$
is bijective. Therefore $X$ is finite, with $|X| = 3$.

**Example 3.14**

**Question.** Prove that the set of natural numbers $\mathbb{N}$ is countable.
**Solution.** The function $f : X \to \mathbb{N}$ defined by
$$f(n) := n\,,$$
is bijective. Therefore $X = \mathbb{N}$ is countable.

**Example 3.15**

**Question.** Let $X$ be the set of even numbers

$$X = \{2n \,:\, n \in \mathbb{N}\}.$$

Prove that $X$ is countable.
**Solution.** Define the map $f : \mathbb{N} \to X$ by

$$f(n) := 2n.$$

We have that:

1. $f$ is injective, because

$$f(m) = f(k) \implies 2m = 2k \quad m = k.$$

2. $f$ is surjective: Suppose that $m \in X$. By definition of $X$, there exists $n \in \mathbb{N}$ such that $m = 2n$. Therefore, $f(n) = m$.

We have shown that $f$ is bijective. Thus, $X$ is countable.

**Example 3.16**

**Question.** Prove that the set of integers $\mathbb{Z}$ is countable.
**Solution.** Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) := \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ -\frac{n+1}{2} & \text{if } n \text{ odd} \end{cases}$$

For example

$$\begin{gathered} f(0) = 0, \quad f(1) = -1, \quad f(2) = 1, \quad f(3) = -2, \\ f(4) = 2, \quad f(5) = -3, \quad f(6) = 3, \quad f(7) = -4. \end{gathered}$$

We have:

1. $f$ is injective: Indeed, suppose that $m \neq n$. If $n$ and $m$ are both even or both odd we have, respectively

$$f(m) = \frac{m}{2} \neq \frac{n}{2} = f(n)$$
$$f(m) = -\frac{m+1}{2} \neq -\frac{n+1}{2} = f(n).$$

   If instead $m$ is even and $n$ is odd, we get

$$f(m) = \frac{m}{2} \neq -\frac{n+1}{2} = f(n),$$

   since the LHS is positive and the RHS is negative. The case when $m$ is odd and $n$ even is similar.

2. $f$ is surjective: Let $z \in \mathbb{Z}$. If $z \geq 0$, then $m := 2z$ belongs to $\mathbb{N}$, is even, and

$$f(m) = f(2z) = z.$$

   If instead $z < 0$, then $m := -2z - 1$ belongs to $\mathbb{N}$, is odd, and

$$f(m) = f(-2z-1) = z.$$

Therefore $f$ is bijective, showing that $\mathbb{Z}$ is countable.

**Proposition 3.17**

Let the set $A_n$ be countable for all $n \in \mathbb{N}$. Define

$$A = \bigcup_{n \in \mathbb{N}} A_n.$$

Then $A$ is countable.

**Theorem 3.18:** $\mathbb{Q}$ is countable

The set of rational numbers $\mathbb{Q}$ is countable.

**Theorem 3.19:** $\mathbb{R}$ is uncountable

The set of Real Numbers $\mathbb{R}$ is **uncountable**.

**Theorem 3.20**

The set of irrational numbers

$$\mathcal{I} := \mathbb{R} \setminus \mathbb{Q}$$

is uncountable.

**Proof**

We know that $\mathbb{R}$ in uncountable and $\mathbb{Q}$ is countable. Suppose by contradiction that $\mathcal{I}$ is countable. Then

$$\mathbb{Q} \cup \mathcal{I}$$

is countable by Proposition 3.17, being union of countable sets. Since by definition

$$\mathbb{R} = \mathbb{Q} \cup \mathcal{I},$$

we conclude that $\mathbb{R}$ is countable. Contradiction.

# 4 Complex Numbers

**Definition 4.1:** Complex Numbers

The set of complex numbers $\mathbb{C}$ is defined as

$$\mathbb{C} := \mathbb{R} + i\mathbb{R} := \{x + iy : x, y \in \mathbb{R}\}.$$

For a complex number

$$z = x + iy \in \mathbb{C}$$

we say that

- $x$ is the **real part** of $z$, and denote it by
$$x = \mathrm{Re}(z)$$
- $y$ is the **imaginary part** of $z$, and denote it by
$$y = \mathrm{Im}(z)$$

We say that

- If $\mathrm{Re}\, z = 0$ then $z$ is a **purely imaginary** number.
- If $\mathrm{Im}\, z = 0$ then $z$ is a **real** number.

**Definition 4.2:** Addition and multiplication in $\mathbb{C}$

Let $z_1, z_2 \in \mathbb{C}$, so that

$$z_1 = x_1 + iy_1 , \quad z_2 = x_2 + iy_2 ,$$

for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$:

1. The sum of $z_1$ and $z_2$ is
$$z_1 + z_2 := (x_1 + x_2) + i\,(y_1 + y_2) .$$
2. The multiplication of $z_1$ and $z_2$ is
$$z_1 \cdot z_2 := (x_1 \cdot x_2 - y_1 \cdot y_2) + i\,(x_1 \cdot y_2 + x_2 \cdot y_1) ,$$

**Example 4.3**

**Question.** Compute $zw$, where

$$z = -2 + 3i , \quad w = 1 - i .$$

**Solution.** Using the definition we compute

$$\begin{aligned} z \cdot w &= (-2 + 3i) \cdot (1 - i) \\ &= (-2 - (-3)) + (2 + 3)i \\ &= 1 + 5i . \end{aligned}$$

Alternatively, we can proceed formally: We just need to recall that $i^2$ has to be replaced with $-1$:

$$\begin{aligned} z \cdot w &= (-2 + 3i) \cdot (1 - i) \\ &= -2 + 2i + 3i - 3i^2 \\ &= (-2 + 3) + (2 + 3)i \\ &= 1 + 5i . \end{aligned}$$

**Proposition 4.4:** Additive inverse in $\mathbb{C}$

The neutral element of addition in $\mathbb{C}$ is the number

$$0 := 0 + 0i .$$

For any $z = x + iy \in \mathbb{C}$, the unique additive inverse is given by

$$-z := -x - iy .$$

**Proposition 4.5:** Multiplicative inverse in $\mathbb{C}$

The neutral element of multiplication in $\mathbb{C}$ is the number

$$1 := 1 + 0i .$$

For any $z = x + iy \in \mathbb{C}$, the unique multiplicative inverse is given by

$$z^{-1} := \frac{x}{x^2 + y^2} + i\,\frac{-y}{x^2 + y^2} .$$

**Proof**

It is immediate to check that 1 is the neutral element of multiplication in $\mathbb{C}$. For the remaining part of the statement, set

$$w := \frac{x}{x^2 + y^2} + i\,\frac{-y}{x^2 + y^2} .$$

We need to check that $z \cdot w = 1$

$$\begin{aligned} z \cdot w &= (x + iy) \cdot \left( \frac{x}{x^2 + y^2} + i\,\frac{-y}{x^2 + y^2} \right) \\ &= \left( \frac{x^2}{x^2 + y^2} - \frac{y \cdot (-y)}{x^2 + y^2} \right) + i \left( \frac{x \cdot (-y)}{x^2 + y^2} + \frac{xy}{x^2 + y^2} \right) \\ &= 1 , \end{aligned}$$

so indeed $z^{-1} = w$.

**Example 4.6**

**Question.** Let $z = 3 + 2i$. Compute $z^{-1}$.
**Solution.** By the formula in Propostion 4.5 we immediately get

$$z^{-1} = \frac{3}{3^2 + 2^2} + \frac{-2}{3^2 + 2^2}\, i = \frac{3}{13} - \frac{2}{13}\, i .$$

Alternatively, we can proceed formally:

$$\begin{aligned}(3+2i)^{-1} &= \frac{1}{3+2i}\\ &= \frac{1}{3+2i}\,\frac{3-2i}{3-2i}\\ &= \frac{3-2i}{3^2+2^2}\\ &= \frac{3}{13}-\frac{2}{13}i\,,\end{aligned}$$

and obtain the same result.

**Theorem 4.7**

$(\mathbb{C},+,\cdot)$ is a field.

**Example 4.8**

**Question.** Let $w=1+i$ and $z=3-i$. Compute $\frac{w}{z}$.
**Solution.** We compute $w/z$ using the two options we have:

1. Using the formula for the inverse from Proposition 4.5 we compute

$$\begin{aligned}z^{-1} &= \frac{x}{x^2+y^2}+i\,\frac{-y}{x^2+y^2}\\ &= \frac{3}{3^2+1^2}-i\,\frac{-1}{3^2+1^2}\\ &= \frac{3}{10}+\frac{1}{10}i\end{aligned}$$

   and therefore

$$\begin{aligned}\frac{w}{z} &= w\cdot z^{-1}\\ &= (1+i)\left(\frac{3}{10}+\frac{1}{10}i\right)\\ &= \left(\frac{3}{10}-\frac{1}{10}\right)+\left(\frac{1}{10}+\frac{3}{10}\right)i\\ &= \frac{2}{10}+\frac{4}{10}i\\ &= \frac{1}{5}+\frac{2}{5}i\end{aligned}$$

2. We proceed formally, using the multiplication by 1 trick. We have

$$\begin{aligned}\frac{w}{z} &= \frac{1+i}{3-i}\\ &= \frac{1+i}{3-i}\frac{3+i}{3+i}\\ &= \frac{3-1+(3+1)i}{3^2+1^2}\\ &= \frac{2}{10}+\frac{4}{10}i\\ &= \frac{1}{5}+\frac{2}{5}i\end{aligned}$$

**Definition 4.9:** Complex conjugate

Let $z=x+iy$. We call the **complex conjugate** of $z$, denoted by $\bar{z}$, the complex number

$$\bar{z}=x-iy\,.$$

**Theorem 4.10**

For all $z_1,z_2\in\mathbb{C}$ it holds:

- $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$
- $\overline{z_1\cdot z_2}=\overline{z_1}\cdot\overline{z_2}$

## 4.1 The complex plane

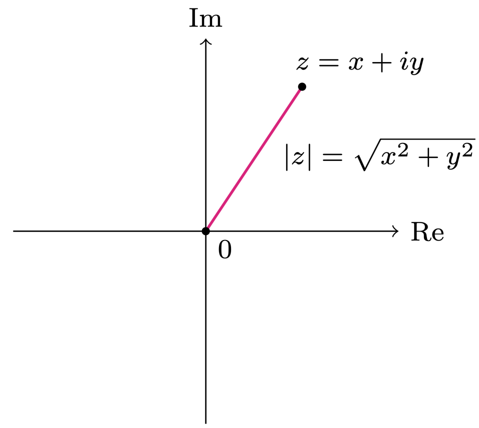


Figure 4.1: A point $z=x+iy\in\mathbb{C}$ can be represented on the complex plane by the point of coordinates $(x,y)$. The distance between $z$ and 0 is given by $|z|=\sqrt{z^2+y^2}$.

**Definition 4.11:** Modulus

The **modulus** of a complex number $z=x+iy$ is defined by

$$|z|:=\sqrt{x^2+y^2}\,.$$

**Definition 4.12:** Distance in $\mathbb{C}$

Given $z_1,z_2\in\mathbb{C}$, we define the **distance** between $z_1$ and $z_2$ as the quantity

$$|z_1-z_2|\,.$$

**Theorem 4.13**

Given $z_1,z_2\in\mathbb{C}$, we have

$$|z_1-z_2|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\,.$$

**Example 4.14**

**Question.** Compute the distance between

$$z=2-4i\,,\quad w=-5+i\,.$$

**Solution.** The distance is

$$\begin{aligned} |z-w| &= |(2-4i)-(-5+i)| \\ &= |7-5i| \\ &= \sqrt{7^2+(-5)^2} \\ &= \sqrt{74} \end{aligned}$$

**Theorem 4.15**

Let $z, z_1, z_2 \in \mathbb{C}$. Then

1. $|z_1 \cdot z_2| = |z_1|\,|z_2|$
2. $|z^n| = |z|^n$ for all $n \in \mathbb{N}$
3. $z \cdot \bar{z} = |z|^2$

**Theorem 4.16:** Triangle inequality in $\mathbb{C}$

For all $x, y, z \in \mathbb{C}$,

1. $|x+y| \leq |x| + |y|$
2. $|x-z| \leq |x-y| + |y-z|$

## 4.2 Polar coordinates

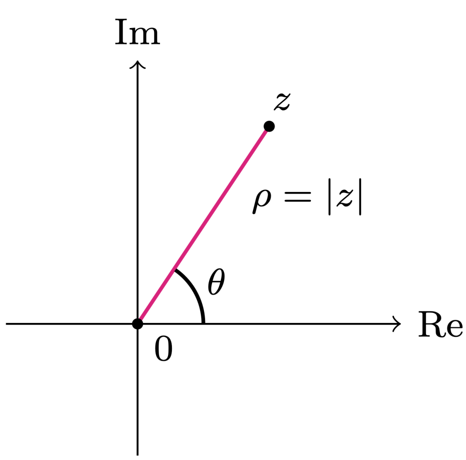


Figure 4.2: Polar coordinates $(\rho, \theta)$ for the complex number $z \in \mathbb{C}$.

**Definition 4.17:** Argument

Let $z \in \mathbb{C}$. The angle $\theta$ between the line connecting the origin and $z$ and the positive real axis is called the **argument** of $z$, and is denoted by

$$\theta := \arg(z)\,.$$

**Example 4.18**

We have the following arguments:

$$\begin{aligned} &\arg(1) = 0 && \arg(i) = \frac{\pi}{2} \\ &\arg(-1) = \pi && \arg(-i) = -\frac{\pi}{2} \\ &\arg(1+i) = \frac{1}{4}\pi && \arg(-1-i) = -\frac{3}{4}\pi \end{aligned}$$

**Theorem 4.19:** Polar coordinates

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$x = \rho\cos(\theta)\,, \quad y = \rho\sin(\theta)\,,$$

where

$$\rho := |z| = \sqrt{x^2+y^2}\,, \quad \theta := \arg(z)\,.$$

**Definition 4.20:** Trigonometric form

Let $z \in \mathbb{C}$. The trigonometric form of $z$ is

$$z = |z|\left[\cos(\theta) + i\sin(\theta)\right]\,,$$

where $\theta = \arg(z)$.

**Example 4.21**

**Question.** Suppose that $z \in \mathbb{C}$ has polar coordinates

$$\rho = \sqrt{8}\,, \quad \theta = \frac{3}{4}\pi\,.$$

Therefore, the trigonometric form of $z$ is

$$z = \sqrt{8}\left[\cos\left(\frac{3}{4}\pi\right) + i\sin\left(\frac{3}{4}\pi\right)\right]\,.$$

Write $z$ in cartesian form.
**Solution.** We have

$$\begin{aligned} x &= \rho\cos(\theta) = \sqrt{8}\cos\left(\frac{3}{4}\pi\right) = -\sqrt{8}\cdot\frac{\sqrt{2}}{2} = -2 \\ y &= \rho\sin(\theta) = \sqrt{8}\sin\left(\frac{3}{4}\pi\right) = \sqrt{8}\cdot\frac{\sqrt{2}}{2} = 2\,. \end{aligned}$$

Therefore, the cartesian form of $z$ is

$$z = x + iy = -2 + 2i\,.$$

**Corollary 4.22:** Computing $\arg(z)$

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$\arg(z) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \end{cases}$$

where arctan is the inverse of tan.

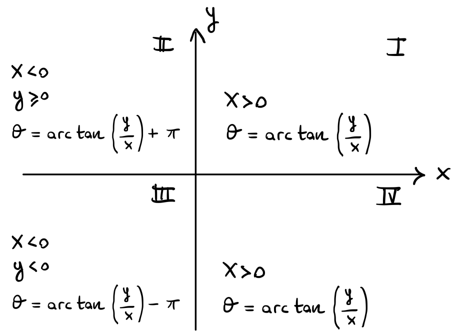


Figure 4.3: The definition of $\arg(z)$ depends on the position of $z$ in the complex plane.

**Example 4.23**

**Question.** Compute the arguments of the complex numbers

$$z = 3 + 4i\,, \quad \bar{z} = 3 - 4i\,, \quad -\bar{z} = -3 + 4i\,, \quad -z = -3 - 4i\,.$$

**Solution.** Using the formula for arg in Corollary 4.22 we have

$$\begin{aligned}
\arg(3 + 4i) &= \arctan\left(\frac{4}{3}\right) \\
\arg(3 - 4i) &= \arctan\left(-\frac{4}{3}\right) = -\arctan\left(\frac{4}{3}\right) \\
\arg(-3 + 4i) &= \arctan\left(-\frac{4}{3}\right) + \pi = -\arctan\left(\frac{4}{3}\right) + \pi \\
\arg(-3 - 4i) &= \arctan\left(\frac{4}{3}\right) - \pi
\end{aligned}$$

## 4.3 Exponential form

**Theorem 4.24:** Euler's identity

For all $\theta \in \mathbb{R}$ it holds

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)\,.$$

**Theorem 4.25**

For all $\theta \in \mathbb{R}$ it holds

$$\left|e^{i\theta}\right| = 1\,.$$

**Theorem 4.26**

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$z = \rho e^{i\theta}\,,$$

where

$$\rho := |z| = \sqrt{x^2 + y^2}\,, \qquad \theta := \arg(z)\,.$$

**Definition 4.27:** Exponential form

The **exponential form** of a complex number $z \in \mathbb{C}$ is

$$z = \rho e^{i\theta} = |z|\, e^{i\arg(z)}\,.$$

**Example 4.28**

**Question.** Write the number

$$z = -2 + 2i$$

in exponential form.
**Solution.** From Example 4.21 we know that $z = -2 + 2i$ can be written in trigonometric form as

$$z = \sqrt{8}\left[\cos\left(\frac{3}{4}\pi\right) + i\sin\left(\frac{3}{4}\pi\right)\right]\,.$$

By Euler's identity we hence obtain the exponential form

$$z = \sqrt{8}e^{i\frac{3}{4}\pi}\,.$$

**Remark 4.29:** Periodicity of exponential

For all $k \in \mathbb{Z}$ we have

$$e^{i\theta} = e^{i(\theta + 2\pi k)}\,, \tag{4.1}$$

meaning that the complex exponential is $2\pi$-periodic.

**Proposition 4.30**

Let $z, z_1, z_2 \in \mathbb{C}$ and suppose that

$$z = \rho e^{i\theta}\,, \quad z_1 = \rho_1 e^{i\theta_1}\,, \quad z_2 = \rho_2 e^{i\theta_2}\,.$$

We have

$$z_1 \cdot z_2 = \rho_1\rho_2 e^{i(\theta_1 + \theta_2)}\,, \quad z^n = \rho^n e^{in\theta}\,,$$

for all $n \in \mathbb{N}$.

**Example 4.31**

**Question.** Compute $(-2 + 2i)^4$.
**Solution.** We have two possibilities:

1. Use the binomial theorem:

$$\begin{aligned}
(-2 + 2i)^4 &= (-2)^4 + \binom{4}{1}(-2)^3 \cdot 2i + \binom{4}{2}(-2)^2 \cdot (2i)^2 \\
&\quad + \binom{4}{3}(-2) \cdot (2i)^3 + (2i)^4 \\
&= 16 - 4 \cdot 8 \cdot 2i - 6 \cdot 4 \cdot 4 + 4 \cdot 2 \cdot 8i + 16 \\
&= 16 - 64i - 96 + 64i + 16 = -64\,.
\end{aligned}$$

2. A much simpler calculation is possible by using the exponential form: We know that

$$-2 + 2i = \sqrt{8}e^{i\frac{3}{4}\pi}$$

by Example 4.28. Hence

$$(-2 + 2i)^4 = \left(\sqrt{8}e^{i\frac{3}{4}\pi}\right)^4 = 8^2 e^{i3\pi} = -64\,,$$

where we used that

$$e^{i3\pi} = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

by $2\pi$ periodicity of $e^{i\theta}$ and Euler's identity.

**Definition 4.32:** Complex exponential

The complex exponential of $z = a + ib \in \mathbb{C}$ is defined as

$$e^z = e^a e^{ib} .$$

**Theorem 4.33**

Let $z, w \in \mathbb{C}$. Then

$$e^{z+w} = e^z e^w , \quad (e^z)^w = e^{zw} . \tag{4.2}$$

**Example 4.34**

**Question.** Compute $i^i$.
**Solution.** We know that

$$|i| = 1 , \quad \arg(i) = \frac{\pi}{2} .$$

Hence we can write $i$ in exponential form

$$i = |i| e^{i \arg(i)} = e^{i\frac{\pi}{2}} .$$

Therefore

$$i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{i^2 \frac{\pi}{2}} = e^{-\frac{\pi}{2}} .$$

## 4.4 Fundamental Theorem of Algebra

**Theorem 4.35:** Fundamental theorem of algebra

Let $p_n(z)$ be a polynomial of degree $n$ with complex coefficients, i.e.,

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0,$$

for some coefficients $a_n, \ldots, a_0 \in \mathbb{C}$ with $a_n \neq 0$. There exist

$$z_1, \ldots, z_n \in \mathbb{C}$$

solutions to the polynomial equation

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0 . \tag{4.3}$$

In particular, $p_n$ factorizes as

$$p_n(z) = a_n (z - z_1)(z - z_2) \cdots (z - z_n) . \tag{4.4}$$

**Example 4.36**

**Question.** Find all the complex solutions to

$$z^2 = -1 \tag{4.5}$$

**Solution.** The equation $z^2 = -1$ is equivalent to

$$p(z) = 0 , \quad p(z) := z^2 + 1 .$$

Since $p$ has degree $n = 2$, the Fundamental Theorem of Algebra tells us that there are two solutions to (4.5). We have already seen that these two solutions are $z = i$ and $z = -i$. Then $p$ factorizes as

$$p(z) = z^2 + 1 = (z - i)(z + i) .$$

**Example 4.37**

**Question.** Find all the complex solutions to

$$z^4 - 1 = 0 . \tag{4.6}$$

**Solution** The associated polynomial equation is

$$p(z) = 0 , \quad p(z) := z^4 - 1 .$$

Since $p$ has degree $n = 4$, the Fundamental Theorem of Algebra tells us that there are 4 solutions to (4.6). Let us find such solutions. We use the well known identity

$$a^2 - b^2 = (a + b)(a - b) , \quad \forall\, a, b \in \mathbb{R} ,$$

to factorize $p$. We get:

$$p(z) = (z^4 - 1) = (z^2 + 1)(z^2 - 1) .$$

We know that

$$z^2 + 1 = 0$$

has solutions $z = \pm i$. Instead

$$z^2 - 1 = 0$$

has solutions $x = \pm 1$. Hence, the four solutions of (4.6) are given by

$$z = 1, -1, i, -i ,$$

and $p$ factorizes as

$$p(z) = z^4 - 1 = (z - 1)(z + 1)(z - i)(z + i) .$$

**Definition 4.38:** Multiplicity

Suppose that the polynomial $p_n$ factorizes as

$$p_n(z) = a_n (z - z_1)^{k_1} (z - z_2)^{k_2} \ldots (z - z_m)^{k_m}$$

with $a_n \neq 0$, $z_1, \ldots, z_m \in \mathbb{C}$ and $k_1, \ldots, k_m \in \mathbb{N}$, $k_i \geq 1$. In this case $p_n$ has degree

$$n = k_1 + \ldots + k_m = \sum_{i=1}^{m} k_i .$$

Note that $z_i$ is solves the equation

$$p_n(z) = 0$$

exactly $k_i$ times. We call $k_i$ the **multiplicity** of the solution $z_i$.

**Example 4.39**

The equation

$$(z - 1)(z - 2)^2 (z + i)^3 = 0$$

has 6 solutions:

- $z = 1$ with multiplicity 1
- $z = 2$ with multiplicity 2
- $z = -i$ with multiplicity 3

## 4.5 Solving polynomial equations

**Proposition 4.40:** Quadratic formula

Let $a, b, c \in \mathbb{R}$, $a \neq 0$ and consider the equation

$$ax^2 + bx + c = 0\,. \tag{4.7}$$

Define

$$\Delta := b^2 - 4ac \in \mathbb{R}\,.$$

The following hold:

1. If $\Delta > 0$ then (4.7) has two distinct real solutions $z_1, z_2 \in \mathbb{R}$ given by
$$z_1 = \frac{-b - \sqrt{\Delta}}{2a}\,, \quad z_2 = \frac{-b + \sqrt{\Delta}}{2a}\,.$$
2. If $\Delta = 0$ then (4.7) has one real solution $z \in \mathbb{R}$ with multiplicity 2. Such solution is given by
$$z = z_1 = z_2 = \frac{-b}{2a}\,.$$
3. If $\Delta < 0$ then (4.7) has two distinct complex solutions $z_1, z_2 \in \mathbb{C}$ given by
$$z_1 = \frac{-b - i\sqrt{-\Delta}}{2a}\,, \quad z_2 = \frac{-b + i\sqrt{-\Delta}}{2a}\,,$$
where $\sqrt{-\Delta} \in \mathbb{R}$, since $-\Delta > 0$.

In all cases, the polynomial at (4.7) factorizes as

$$az^2 + bz + c = a(z - z_1)(z - z_2)\,.$$

**Example 4.41**

**Question.** Solve the following equations:

1. $3z^2 - 6z + 2 = 0$
2. $4z^2 - 8z + 4 = 0$
3. $z^2 + 2z + 3 = 0$

**Solution.**

1. We have that
$$\Delta = (-6)^2 - 4 \cdot 3 \cdot 2 = 12 > 0$$
Therefore the equation has two distinct real solutions, given by
$$z = \frac{-(-6) \pm \sqrt{12}}{2 \cdot 3} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{\sqrt{3}}{3}$$
In particular we have the factorization
$$3z^2 - 6z + 2 = 3\left(z - 1 - \frac{\sqrt{3}}{3}\right)\left(z - 1 + \frac{\sqrt{3}}{3}\right)\,.$$
2. We have that
$$\Delta = (-8)^2 - 4 \cdot 4 \cdot 4 = 0\,.$$
Therefore there exists one solution with multiplicity 2. This is given by
$$z = \frac{-(-8)}{2 \cdot 4} = 1\,.$$
In particular we have the factorization
$$4z^2 - 8x + 4 = 4(z - 1)^2\,.$$
3. We have
$$\Delta = 2^2 - 4 \cdot 1 \cdot 3 = -8 < 0\,.$$
Therefore there are two complex solutions given by
$$z = \frac{-2 \pm i\sqrt{8}}{2 \cdot 1} = -1 \pm i\sqrt{2}\,.$$
In particular we have the factorization
$$z^2 + 2z + 3 = (z + 1 - i\sqrt{2})(z + 1 + i\sqrt{2})\,.$$

**Proposition 4.42:** Quadratic formula with complex coefficients

Let $a, b, c \in \mathbb{C}$, $a \neq 0$. The two solutions to

$$az^2 + bz + c = 0$$

are given by

$$z_1 = \frac{-b + S_1}{2a}\,, \quad z_2 = \frac{-b + S_2}{2a}\,,$$

where $S_1$ and $S_2$ are the two solutions to

$$z^2 = \Delta\,, \quad \Delta := b^2 - 4ac\,.$$

**Example 4.43**

**Question** Find all the solutions to

$$\frac{1}{2}z^2 - (3 + i)z + (4 - i) = 0\,. \tag{4.8}$$

**Solution.** We have

$$\begin{aligned}\Delta &= (-(3 + i))^2 - 4 \cdot \frac{1}{2} \cdot (4 - i) \\ &= 8 + 6i - 8 + 2i \\ &= 8i\,.\end{aligned}$$

Therefore $\Delta \in \mathbb{C}$. We have to find solutions $S_1$ and $S_2$ to the equation

$$z^2 = \Delta = 8i\,. \tag{4.9}$$

We look for solutions of the form $z = a + ib$. Then we must have that

$$z^2 = (a + ib)^2 = a^2 - b^2 + 2abi = 8i\,.$$

Thus

$$a^2 - b^2 = 0\,, \quad 2ab = 8\,.$$

From the first equation we conclude that $|a| = |b|$. From the second equation we have that $ab = 4$, and therefore $a$ and $b$ must have the same sign. Hence $a = b$, and

$$2ab = 8 \quad \Longrightarrow \quad a = b = \pm 2\,.$$

From this we conclude that the solutions to (4.9) are

$$S_1 = 2 + 2i\,, \quad S_2 = -2 - 2i\,.$$

Hence the solutions to (4.8) are

$$\begin{aligned}z_1 &= \frac{3 + i + S_1}{2 \cdot \frac{1}{2}} = 3 + i + S_1 \\ &= 3 + i + 2 + 2i = 5 + 3i\,,\end{aligned}$$

and

$$z_2 = \frac{3+i+S_2}{2\cdot\frac{1}{2}} = 3+i+S_2$$
$$= 3+i-2-2i = 1-i\,.$$

In particular, the given polynomial factorizes as

$$\frac{1}{2}z^2-(3+i)z+(4-i) = \frac{1}{2}(z-z_1)(z-z_2)$$
$$= \frac{1}{2}(z-5-3i)(z-1+i)\,.$$

**Example 4.44**

**Question.** Consider the equation

$$z^3-7z^2+6z=0\,.$$

1. Check whether $z=0,1,-1$ are solutions.
2. Using your answer from Point 1, and polynomial division, find all the solutions.

**Solution.**

1. By direct inspection we see that $z=0$ and $z=1$ are solutions.
2. Since $z=0$ is a solution, we can factorize

   $$z^3-7z^2+6z = z\left(z^2-7z+6\right)\,.$$

   We could now use the quadratic formula on the term $z^2-7z+6$ to find the remaining two roots. However, we have already observed that $z=1$ is a solution. Therefore $z-1$ divides $z^2-7z+6$. Using polynomial long division, see Figure 4.4, we find that

   $$\frac{z^2-7z+6}{z-1} = z-6\,.$$

   Therefore the last solution is $z=6$, and

   $$z^3-7z^2+6z = z(z-1)(z-6)\,.$$

$$\begin{array}{r|l}
 & z-6 \\ \hline
z-1 & z^2-7z+6 \\
 & -z^2+z \\ \hline
 & -6z+6 \\
 & 6z-6 \\ \hline
 & 0
\end{array}$$

Figure 4.4: Polynomial long division between $z^2-7z+6$ and $z-1$.

**Example 4.45**

**Question.** Find all the complex solutions to

$$z^3-7z+6=0\,.$$

**Solution.** It is easy to see $z=1$ is a solution. This means that $z-1$ divides $z^3-7z+6$. By using polynomial long division, see Figure 4.5, we compute that

$$\frac{z^3-7z+6}{z-1} = z^2+z-6\,.$$

We are now left to solve

$$z^2+z-6=0\,.$$

Using the quadratic formula, we see that the above is solved by $z=2$ and $z=-3$. Therefore the given polynomial factorizes as

$$z^3-7z+6 = (z-1)(z-2)(z+3)\,.$$

$$\begin{array}{r|l}
 & z^2+z-6 \\ \hline
z-1 & z^3-7z+6 \\
 & -z^3+z^2 \\ \hline
 & z^2-7z \\
 & -z^2+z \\ \hline
 & -6z+6 \\
 & 6z-6 \\ \hline
 & 0
\end{array}$$

Figure 4.5: Polynomial long division between $z^3-7z+6$ and $z-1$.

## 4.6 Roots of unity

**Theorem 4.46**

Let $n\in\mathbb{N}$ and consider the equation

$$z^n=1\,. \tag{4.10}$$

All the $n$ solutions to (4.10) are given by

$$z_k = \exp\left(i\frac{2\pi k}{n}\right)\,, \quad k=0,\dots,n-1\,,$$

where $\exp(x)$ denotes $e^x$.

**Definition 4.47**

The $n$ solutions to

$$z^n=1$$

are called the **roots of unity**.

**Example 4.48**

**Question.** Find all the solutions to

$$z^4=1\,.$$

**Solution.** The 4 solutions are given by

$$z_k = \exp\left(i\frac{2\pi k}{4}\right) = \exp\left(i\frac{\pi k}{2}\right)\,,$$

for $k=0,1,2,3$. We compute:

$$z_0 = e^{i0}=1\,, \qquad z_1 = e^{i\frac{\pi}{2}} = i\,,$$
$$z_2 = e^{i\pi} = -1\,, \qquad z_3 = e^{i\frac{3\pi}{2}} = -i\,.$$

Note that for $k=4$ we would again get the solution $z=e^{i2\pi}=1$.

**Example 4.49**

**Question.** Find all the solutions to

$$z^3 = 1\,.$$

**Solution.** The 3 solutions are given by

$$z_k = \exp\left(i\frac{2\pi k}{3}\right),$$

for $k = 0, 1, 2$. We compute:

$$z_0 = e^{i0} = 1, \quad z_1 = e^{i\frac{2\pi}{3}}, \quad z_2 = e^{i\frac{4\pi}{3}}.$$

We can write $z_1$ and $z_2$ in cartesian form:

$$z_1 = e^{i\frac{2\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

and

$$z_2 = e^{i\frac{4\pi}{3}} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i\,.$$

## 4.7 Roots in $\mathbb{C}$

**Theorem 4.50**

Let $n \in \mathbb{N}$, $c \in \mathbb{C}$ and consider the equation

$$z^n = c\,. \tag{4.11}$$

All the $n$ solutions to (4.11) are given by

$$z_k = \sqrt[n]{|c|}\,\exp\left(i\,\frac{\theta + 2\pi k}{n}\right), \quad k = 0, \dots, n-1\,,$$

where $\sqrt[n]{|c|}$ is the $n$-th root of the real number $|c|$, and $\theta = \arg(c)$.

**Example 4.51**

**Question.** Find all the $z \in \mathbb{C}$ such that

$$z^5 = -32\,.$$

**Solution.** Let $c = -32$. We have

$$|c| = |-32| = 32 = 2^5\,, \quad \theta = \arg(-32) = \pi\,.$$

The 5 solutions are given by

$$z_k = (2^5)^{\frac{1}{5}}\exp\left(i\pi\,\frac{1+2k}{5}\right), \quad k \in \mathbb{Z}\,,$$

for $k = 0, 1, 2, 3, 4$. We get

$$z_0 = 2e^{i\frac{\pi}{5}} \qquad z_1 = 2e^{i\frac{3\pi}{5}}$$
$$z_2 = 2e^{i\pi} = -2 \qquad z_3 = 2e^{i\frac{7\pi}{5}}$$
$$z_4 = 2e^{i\frac{9\pi}{5}}$$

**Example 4.52**

**Question.** Find all the $z \in \mathbb{C}$ such that

$$z^4 = 9\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right).$$

**Solution.** Set

$$c := 9\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right).$$

The complex number $c$ is already in the trigonometric form, so that we can immediately obtain

$$|c| = 9\,, \quad \theta = \arg(c) = \frac{\pi}{3}\,.$$

The 4 solutions are given by

$$\begin{aligned} z_k &= \sqrt[4]{9}\,\exp\left(i\,\frac{\pi/3 + 2\pi k}{4}\right) \\ &= \sqrt{3}\exp\left(i\pi\,\frac{1+6k}{12}\right) \end{aligned}$$

for $k = 0, 1, 2, 3$. We compute

$$z_0 = \sqrt{3}e^{i\pi\frac{1}{12}} \qquad z_1 = \sqrt{3}e^{i\pi\frac{7}{12}}$$
$$z_2 = \sqrt{3}e^{i\pi\frac{13}{12}} \qquad z_3 = \sqrt{3}e^{i\pi\frac{19}{12}}$$

# 5 Sequences in $\mathbb{R}$

**Definition 5.1:** Convergent sequence

The real sequence $(a_n)$ **converges** to $a$, or equivalently has limit $a$, denoted by

$$\lim_{n\to\infty} a_n = a\,,$$

if for all $\varepsilon \in \mathbb{R}, \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}, n \geq N$ it holds that

$$|a_n - a| < \varepsilon\,.$$

Using quantifiers, we can write this as

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ s.t. } \forall\, n \geq N\,,\ |a_n - a| < \varepsilon\,.$$

The sequence $(a_n)_{n\in\mathbb{N}}$ is **convergent** if it admits limit.

**Theorem 5.2**

Let $p > 0$. Then

$$\lim_{n\to\infty} \frac{1}{n^p} = 0\,.$$

**Proof**

Let $p > 0$. We have to show that

$$\forall \varepsilon > 0\,, \exists\, N \in \mathbb{N} \text{ s.t. } \forall\, n \geq N\,, \left|\frac{1}{n^p} - 0\right| < \varepsilon\,.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\varepsilon^{1/p}}\,. \tag{5.1}$$

Let $n \geq N$. Since $p > 0$, we have $n^p \geq N^p$, which implies

$$\frac{1}{n^p} \leq \frac{1}{N^p}\,.$$

By (5.1) we deduce

$$\frac{1}{N^p} < \varepsilon\,.$$

Then

$$\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon\,.$$

**Example 5.3**

**Question.** Using the definition of convergence, prove that

$$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}\,.$$

**Solution.**

1. *Rough Work:* Let $\varepsilon > 0$. We want to find $N \in \mathbb{N}$ such that

$$\left|\frac{n}{2n+3} - \frac{1}{2}\right| < \varepsilon\,, \quad \forall\, n \geq N\,.$$

To this end, we compute:

$$\left|\frac{n}{2n+3} - \frac{1}{2}\right| = \left|\frac{-3}{4n+6}\right| = \frac{3}{4n+6}\,.$$

Therefore

$$\left|\frac{n}{2n+3} - \frac{1}{2}\right| < \varepsilon \iff \frac{3}{4n+6} < \varepsilon \iff n > \frac{3}{4\varepsilon} - \frac{6}{4}\,.$$

Looking at the above equivalences, it is clear that $N \in \mathbb{N}$ has to be chosen so that

$$N > \frac{3}{4\varepsilon} - \frac{6}{4}\,. \tag{5.2}$$

2. *Formal Proof:* We have to show that

$$\forall \varepsilon > 0\,, \exists\, N \in \mathbb{N} \text{ s.t. } \forall\, n \geq N\,, \left|\frac{n}{2n+3} - \frac{1}{2}\right| < \varepsilon\,.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that (5.2) holds. By the rough work shown above, inequality (5.2) is equivalent to

$$\frac{3}{4N+6} < \varepsilon\,.$$

Let $n \geq N$. Then

$$\left|\frac{n}{2n+3} - \frac{1}{2}\right| = \frac{3}{4n+6} \leq \frac{3}{4N+6} < \varepsilon\,,$$

where in the third line we used that $n \geq N$.

**Theorem 5.4:** Uniqueness of limit

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Suppose that

$$\lim_{n\to\infty} a_n = a\,, \quad \lim_{n\to\infty} a_n = b\,.$$

Then $a = b$.

## 5.1 Divergent sequences

**Definition 5.5:** Divergent sequence

We say that a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is **divergent** if it is not convergent.

**Theorem 5.6**

Let $(a_n)$ be the sequence defined by

$$a_n = (-1)^n\,.$$

Then $(a_n)$ does not converge.

**Proof**

Suppose by contradiction that $a_n \to a$ for some $a \in \mathbb{R}$. Let

$$\varepsilon := \frac{1}{2}.$$

Since $a_n \to a$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon = \frac{1}{3} \quad \forall\, n \geq N.$$

If we take $n = 2N$, then $n \geq N$ and

$$|a_{2N} - a| = |1 - a| < \frac{1}{2}.$$

If we take $n = 2N+1$, then $n \geq N$ and

$$|a_{2N+1} - a| = |-1 - a| < \frac{1}{2}.$$

Therefore

$$\begin{aligned} 2 &= |(1-a) - (-1-a)| \\ &\leq |1-a| + |-1-a| \\ &< \frac{1}{2} + \frac{1}{2} = 1, \end{aligned}$$

which is a contradiction. Hence $(a_n)$ is divergent.

## 5.2 Bounded sequences

**Definition 5.7:** Bounded sequence

A sequence $(a_n)_{n\in\mathbb{N}}$ is called **bounded** if there exists a constant $M \in \mathbb{R}$, with $M > 0$, such that

$$|a_n| \leq M, \quad \forall\, n \in \mathbb{N}.$$

**Theorem 5.8**

Every convergent sequence is bounded.

**Example 5.9**

The sequence

$$a_n = (-1)^n$$

is bounded ($M = 1$) but not convergent.

**Corollary 5.10**

If a sequence is not bounded, then the sequence does not converge.

**Remark 5.11**

For a sequence $(a_n)$ to be unbounded, it means that

$$\forall\, M > 0, \ \exists\, n \in \mathbb{N} \ \text{ s.t. } \ |a_n| > M.$$

**Theorem 5.12**

Let $p > 0$. The sequence $a_n = n^p$ is unbounded, and hence divergent.

**Theorem 5.13**

The sequence $a_n = \log n$ is unbounded, and hence divergent.

## 5.3 Algebra of limits

**Theorem 5.14:** Algebra of limits

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$. Suppose that

$$\lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} b_n = b,$$

for some $a, b \in \mathbb{R}$. Then,

1. Limit of sum is the sum of limits:

$$\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$$

2. Limit of product is the product of limits:

$$\lim_{n\to\infty} (a_n b_n) = ab$$

3. If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$$

**Example 5.15**

**Question.** Prove that

$$\lim_{n\to\infty} \frac{3n}{7n+4} = \frac{3}{7}.$$

**Solution.** We can rewrite

$$\frac{3n}{7n+4} = \frac{3}{7 + \frac{4}{n}}$$

From Theorem 5.2, we know that

$$\frac{1}{n} \to 0.$$

Hence, it follows from Theorem 5.14 Point 2 that

$$\frac{4}{n} = 4 \cdot \frac{1}{n} \to 4 \cdot 0 = 0.$$

By Theorem 5.14 Point 1 we have

$$7 + \frac{4}{n} \to 7 + 0 = 7.$$

Finally we can use Theorem 5.14 Point 3 to infer

$$\frac{3n}{7n+4} = \frac{3}{7 + \frac{4}{n}} \to \frac{3}{7}.$$

**Example 5.16**

**Question.** Prove that

$$\lim_{n\to\infty} \frac{n^2-1}{2n^2-3} = \frac{1}{2}\,.$$

**Solution.** Factor $n^2$ to obtain

$$\frac{n^2-1}{2n^2-3} = \frac{1-\dfrac{1}{n^2}}{2-\dfrac{3}{n^2}}\,.$$

By Theorem 5.2 we have

$$\frac{1}{n^2} \to 0\,.$$

We can then use the Algebra of Limits Theorem 5.14 Point 2 to infer

$$\frac{3}{n^2} \to 3 \cdot 0 = 0$$

and Theorem 5.14 Point 1 to get

$$1 - \frac{1}{n^2} \to 1 - 0 = 1\,, \quad 2 - \frac{3}{n^2} \to 2 - 0 = 2\,.$$

Finally we use Theorem 5.14 Point 3 and conclude

$$\frac{1-\dfrac{1}{n^2}}{2-\dfrac{3}{n^2}} \to \frac{1}{2}\,.$$

Therefore

$$\lim_{n\to\infty} \frac{n^2-1}{2n^2-3} = \lim_{n\to\infty} \frac{1-\dfrac{1}{n^2}}{2-\dfrac{3}{n^2}} = \frac{1}{2}\,.$$

**Example 5.17**

**Question.** Prove that the sequence

$$a_n = \frac{4n^3+8n+1}{7n^2+2n+1}$$

does not converge.

**Solution.** To show that the sequence $(a_n)$ does not converge, we divide by the largest power in the denominator, which in this case is $n^2$

$$\begin{aligned} a_n &= \frac{4n^3+8n+1}{7n^2+2n+1} \\ &= \frac{4n+\dfrac{8}{n}+\dfrac{1}{n^2}}{7+\dfrac{2}{n}+\dfrac{1}{n^2}} = \frac{b_n}{c_n}\,, \end{aligned}$$

where we set

$$b_n := 4n + \frac{8}{n} + \frac{1}{n^2}\,, \quad c_n := 7 + \frac{2}{n} + \frac{1}{n^2}\,.$$

Using the Algebra of Limits Theorem 5.14 we see that

$$c_n = 7 + \frac{2}{n} + \frac{1}{n^2} \to 7\,.$$

Suppose by contradiction that

$$a_n \to a$$

for some $a \in \mathbb{R}$. Then, by the Algebra of Limits, we would infer

$$b_n = c_n \cdot a_n \to 7a\,,$$

concluding that $b_n$ is convergent to $7a$. We have that

$$b_n = 4n + d_n\,, \quad d_n := \frac{8}{n} + \frac{1}{n^2}\,.$$

Again by the Algebra of Limits Theorem 5.14 we get that

$$d_n = \frac{8}{n} + \frac{1}{n^2} \to 0\,,$$

and hence

$$4n = b_n - d_n \to 7a - 0 = 7a\,.$$

This is a contradiction, since the sequence $(4n)$ is unbounded, and hence cannot be convergent. Hence $(a_n)$ is not convergent.

**Example 5.18**

**Question.** Define the sequence

$$a_n := \frac{2n^3+7n+1}{5n+9} \cdot \frac{8n+9}{6n^3+8n^2+3}\,.$$

Prove that

$$\lim_{n\to\infty} a_n = \frac{8}{15}\,.$$

**Solution.** The first fraction in $(a_n)$ does not converge, as it is unbounded. Therefore we cannot use Point 2 in Theorem 5.14 directly. However, we note that

$$\begin{aligned} a_n &= \frac{2n^3+7n+1}{5n+9} \cdot \frac{8n+9}{6n^3+8n^2+3} \\ &= \frac{8n+9}{5n+9} \cdot \frac{2n^3+7n+1}{6n^3+8n^2+3}\,. \end{aligned}$$

Factoring out $n$ and $n^3$, respectively, and using the Algebra of Limits, we see that

$$\frac{8n+9}{5n+9} = \frac{8+9/n}{5+9/n} \to \frac{8+0}{5+0} = \frac{8}{5}$$

and

$$\frac{2+7/n^2+1/n^3}{6+8/n+3/n^3} \to \frac{2+0+0}{6+0+0} = \frac{1}{3}$$

Therefore Theorem 5.14 Point 2 ensures that

$$a_n \to \frac{8}{5} \cdot \frac{1}{3} = \frac{8}{15}\,.$$

**Example 5.19**

**Question.** Prove that

$$a_n = \frac{n^{7/3}+2\sqrt{n}+7}{4n^{3/2}+5n}$$

does not converge.

**Solution.** The largest power of $n$ in the denominator is $n^{3/2}$. Hence we factor out $n^{3/2}$

$$\begin{aligned} a_n &= \frac{n^{7/3} + 2\sqrt{n} + 7}{4n^{3/2} + 5n} \\ &= \frac{n^{7/3-3/2} + 2n^{1/2-3/2} + 7n^{-3/2}}{4 + 5n^{-3/2}} \\ &= \frac{n^{5/6} + 2n^{-1} + 7n^{-3/2}}{4 + 5n^{-3/2}} \\ &= \frac{b_n}{c_n} \end{aligned}$$

where we set

$$b_n := n^{5/6} + 2n^{-1} + 7n^{-3/2}, \quad c_n := 4 + 5n^{-3/2}.$$

We see that $b_n$ is unbounded while $c_n \to 4$. By the Algebra of Limits (and usual contradiction argument) we conclude that $(a_n)$ is divergent.

**Theorem 5.20**

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that

$$\lim_{n\to\infty} a_n = a,$$

for some $a \in \mathbb{R}$. If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $a \geq 0$, then

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}.$$

**Example 5.21**

**Question.** Define the sequence

$$a_n = \sqrt{9n^2 + 3n + 1} - 3n.$$

Prove that

$$\lim_{n\to\infty} a_n = \frac{1}{2}.$$

**Solution.** We first rewrite

$$\begin{aligned} a_n &= \sqrt{9n^2 + 3n + 1} - 3n \\ &= \frac{\left(\sqrt{9n^2 + 3n + 1} - 3n\right)\left(\sqrt{9n^2 + 3n + 1} + 3n\right)}{\sqrt{9n^2 + 3n + 1} + 3n} \\ &= \frac{9n^2 + 3n + 1 - (3n)^2}{\sqrt{9n^2 + 3n + 1} + 3n} \\ &= \frac{3n + 1}{\sqrt{9n^2 + 3n + 1} + 3n}. \end{aligned}$$

The biggest power of $n$ in the denominator is $n$. Therefore we factor out $n$:

$$\begin{aligned} a_n &= \sqrt{9n^2 + 3n + 1} - 3n \\ &= \frac{3n + 1}{\sqrt{9n^2 + 3n + 1} + 3n} \\ &= \frac{3 + \frac{1}{n}}{\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 3}. \end{aligned}$$

By the Algebra of Limits we have

$$9 + \frac{3}{n} + \frac{1}{n^2} \to 9 + 0 + 0 = 9.$$

Therefore we can use Theorem 5.20 to infer

$$\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} \to \sqrt{9}.$$

By the Algebra of Limits we conclude:

$$a_n = \frac{3 + \frac{1}{n}}{\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 3} \to \frac{3 + 0}{\sqrt{9} + 3} = \frac{1}{2}.$$

**Example 5.22**

**Question.** Prove that the sequence

$$a_n = \sqrt{9n^2 + 3n + 1} - 2n$$

does not converge.
**Solution.** We rewrite $a_n$ as

$$\begin{aligned} a_n &= \sqrt{9n^2 + 3n + 1} - 2n \\ &= \frac{(\sqrt{9n^2 + 3n + 1} - 2n)(\sqrt{9n^2 + 3n + 1} + 2n)}{\sqrt{9n^2 + 3n + 1} + 2n} \\ &= \frac{9n^2 + 3n + 1 - (2n)^2}{\sqrt{9n^2 + 3n + 1} + 2n} \\ &= \frac{5n^2 + 3n + 1}{\sqrt{9n^2 + 3n + 1} + 2n} \\ &= \frac{5n + 3 + \frac{1}{n}}{\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 2} \\ &= \frac{b_n}{c_n}, \end{aligned}$$

where we factored $n$, being it the largest power of $n$ in the denominator, and defined

$$b_n := 5n + 3 + \frac{1}{n}, \quad c_n := \sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 2.$$

Note that

$$9 + \frac{3}{n} + \frac{1}{n^2} \to 9$$

by the Algebra of Limits. Therefore

$$\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} \to \sqrt{9} = 3$$

by Theorem 5.20. Hence

$$c_n = \sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 2 \to 3 + 2 = 5.$$

The numerator

$$b_n = 5n + 3 + \frac{1}{n}$$

is instead unbounded. Therefore $(a_n)$ is not convergent, by the Algebra of Limits and the usual contradiction argument.

## 5.4 Limit Tests

**Theorem 5.23:** Squeeze theorem

Let $(a_n)$, $(b_n)$ and $(c_n)$ be sequences in $\mathbb{R}$. Suppose that

$$b_n \le a_n \le c_n\,, \quad \forall\, n \in \mathbb{N}\,,$$

and that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L\,.$$

Then

$$\lim_{n\to\infty} a_n = L\,.$$

**Example 5.24**

**Question.** Prove that

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0\,.$$

**Solution.** For all $n \in \mathbb{N}$ we can estimate

$$-1 \le (-1)^n \le 1\,.$$

Therefore

$$\frac{-1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}\,, \quad \forall\, n \in \mathbb{N}\,.$$

Moreover

$$\lim_{n\to\infty} \frac{-1}{n} = -1 \cdot 0 = 0\,, \quad \lim_{n\to\infty} \frac{1}{n} = 0\,.$$

By the Squeeze Theorem 5.23 we conclude

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0\,.$$

**Example 5.25**

**Question.** Prove that

$$\lim_{n\to\infty} \frac{\cos(3n) + 9n^2}{11n^2 + 15\sin(17n)} = \frac{9}{11}\,.$$

**Solution.** We know that

$$-1 \le \cos(x) \le 1\,, \quad -1 \le \sin(x) \le 1\,, \quad \forall\, x \in \mathbb{R}\,.$$

Therefore, for all $n \in \mathbb{N}$

$$-1 \le \cos(3n) \le 1\,, \quad -1 \le \sin(17n) \le 1\,.$$

We can use the above to estimate the numerator in the given sequence:

$$-1 + 9n^2 \le \cos(3n) + 9n^2 \le 1 + 9n^2\,. \tag{5.3}$$

Concerning the denominator, we have

$$11n^2 - 15 \le 11n^2 + 15\sin(17n) \le 11n^2 + 15$$

and therefore

$$\frac{1}{11n^2 + 15} \le \frac{1}{11n^2 + 15\sin(17n)} \le \frac{1}{11n^2 - 15}\,. \tag{5.4}$$

Putting together (5.3)-(5.4) we obtain

$$\frac{-1 + 9n^2}{11n^2 + 15} \le \frac{\cos(3n) + 9n^2}{11n^2 + 15\sin(17n)} \le \frac{1 + 9n^2}{11n^2 - 15}\,.$$

By the Algebra of Limits we infer

$$\frac{-1 + 9n^2}{11n^2 + 15} = \frac{-\frac{1}{n^2} + 9}{11 + \frac{15}{n^2}} \to \frac{0 + 9}{11 + 0} = \frac{9}{11}$$

and

$$\frac{1 + 9n^2}{11n^2 - 15} = \frac{\frac{1}{n^2} + 9}{11 - \frac{15}{n^2}} \to \frac{0 + 9}{11 + 0} = \frac{9}{11}\,.$$

Applying the Squeeze Theorem 5.23 we conclude

$$\lim_{n\to\infty} \frac{\cos(3n) + 9n^2}{11n^2 + 15\sin(17n)} = \frac{9}{11}\,.$$

**Definition 5.26:** Geometric sequence

A sequence $(a_n)$ is called a **geometric sequence** if

$$a_n = x^n\,,$$

for some $x \in \mathbb{R}$.

**Theorem 5.27:** Geometric Sequence Test

Let $x \in \mathbb{R}$ and let $(a_n)$ be the sequence defined by $a_n := x^n$. We have:

1. If $|x| < 1$, then
$$\lim_{n\to\infty} a_n = 0\,.$$
2. If $|x| > 1$, then sequence $(a_n)$ is unbounded, and hence divergent.

**Example 5.28**

We can apply Theorem 5.27 to prove convergence or divergence for the following sequences.

1. We have
$$\left(\frac{1}{2}\right)^n \longrightarrow 0$$
since
$$\left|\frac{1}{2}\right| = \frac{1}{2} < 1\,.$$
2. We have
$$\left(\frac{-1}{2}\right)^n \longrightarrow 0$$
since
$$\left|\frac{-1}{2}\right| = \frac{1}{2} < 1\,.$$
3. The sequence
$$a_n = \left(\frac{-3}{2}\right)^n$$
does not converge, since
$$\left|\frac{-3}{2}\right| = \frac{3}{2} > 1\,.$$

4. As $n \to \infty$,
$$\frac{3^n}{(-5)^n} = \left(-\frac{3}{5}\right)^n \longrightarrow 0$$
since
$$\left|-\frac{3}{5}\right| = \frac{3}{5} < 1\,.$$

5. The sequence
$$a_n = \frac{(-7)^n}{2^{2n}}$$
does not converge, since
$$\frac{(-7)^n}{2^{2n}} = \frac{(-7)^n}{\left(2^2\right)^n} = \left(-\frac{7}{4}\right)^n$$
and
$$\left|-\frac{7}{4}\right| = \frac{7}{4} > 1\,.$$

**Theorem 5.29:** Ratio Test

Let $(a_n)$ be a sequence in $\mathbb{R}$ such that
$$a_n \neq 0\,, \quad \forall\, n \in \mathbb{N}\,.$$

1. Suppose that the following limit exists:
$$L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|\,.$$
Then,
   - If $L < 1$ we have
   $$\lim_{n\to\infty} a_n = 0\,.$$
   - If $L > 1$, the sequence $(a_n)$ is unbounded, and hence does not converge.

2. Suppose that there exists $N \in \mathbb{N}$ and $L > 1$ such that
$$\left|\frac{a_{n+1}}{a_n}\right| \geq L\,, \quad \forall\, n \geq N\,.$$
Then the sequence $(a_n)$ is unbounded, and hence does not converge.

**Example 5.30**

**Question.** Prove that
$$a_n = \frac{3^n}{n!} \to 0\,,$$
where we recall that $n!$ (pronounced $n$ factorial) is defined by
$$n! := n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1\,.$$

**Solution.** We have
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left(\frac{3^{n+1}}{(n+1)!}\right)}{\left(\frac{3^n}{n!}\right)} = \frac{3}{n+1} \longrightarrow L = 0\,.$$
Hence, $L = 0 < 1$ so $a_n \to 0$ by the Ratio Test in Theorem 5.29.

**Example 5.31**

**Question.** Prove that the sequence is divergent
$$a_n = \frac{n! \cdot 3^n}{\sqrt{(2n)!}}\,.$$

**Solution.** We have
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)! \cdot 3^{n+1}}{\sqrt{(2(n+1))!}} \frac{\sqrt{(2n)!}}{n! \cdot 3^n} \\ &= \frac{(n+1)!}{n!} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{\sqrt{(2n)!}}{\sqrt{(2(n+1))!}}\end{aligned}$$
For the first two fractions we have
$$\frac{(n+1)!}{n!} \cdot \frac{3^{n+1}}{3^n} = 3(n+1)\,,$$
while for the third fraction
$$\begin{aligned}\frac{\sqrt{(2n)!}}{\sqrt{(2(n+1))!}} &= \sqrt{\frac{(2n)!}{(2n+2)!}} \\ &= \sqrt{\frac{(2n)!}{(2n+2)\cdot(2n+1)\cdot(2n)!}} \\ &= \frac{1}{\sqrt{(2n+1)(2n+2)}}\,.\end{aligned}$$
Therefore, using the Algebra of Limits,
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{3(n+1)}{\sqrt{(2n+1)(2n+2)}} \\ &= \frac{3n\left(1+\frac{1}{n}\right)}{\sqrt{n^2\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}} \\ &= \frac{3\left(1+\frac{1}{n}\right)}{\sqrt{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}} \longrightarrow \frac{3}{\sqrt{4}} = \frac{3}{2} > 1\,.\end{aligned}$$
By the Ratio Test we conclude that $(a_n)$ is divergent.

**Example 5.32**

**Question.** Prove that the following sequence is divergent
$$a_n = \frac{n!}{100^n}\,.$$

**Solution.** We have
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{100^n}{100^{n+1}} \frac{(n+1)!}{n!} = \frac{n+1}{100}\,.$$
Choose $N = 101$. Then for all $n \geq N$,
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{100} \geq \frac{N+1}{100} = \frac{101}{100} > 1\,.$$
Hence $a_n$ is divergent by the Ratio Test.

## 5.5 Monotone sequences

**Definition 5.33:** Monotone sequence

Let $(a_n)$ be a real sequence. We say that:

1. $(a_n)$ is **increasing** if
$$a_n \leq a_{n+1}, \quad \forall\, n \geq N.$$
2. $(a_n)$ is **decreasing** if
$$a_n \geq a_{n+1}, \quad \forall\, n \geq N.$$
3. $(a_n)$ is **monotone** if it is either increasing or decreasing.

**Example 5.34**

**Question.** Prove that the following sequence is increasing

$$a_n = \frac{n-1}{n}.$$

**Solution.** We have

$$a_{n+1} = \frac{n}{n+1} > \frac{n-1}{n} = a_n,$$

where the inequality holds because

$$\begin{aligned} \frac{n}{n+1} > \frac{n-1}{n} \quad & \iff \quad n^2 > (n-1)(n+1) \\ & \iff \quad n^2 > n^2 - 1 \\ & \iff \quad 0 > -1 \end{aligned}$$

**Example 5.35**

**Question.** Prove that the following sequence is decreasing

$$a_n = \frac{1}{n}.$$

**Solution.** We immediately see that

$$a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}.$$

**Theorem 5.36:** Monotone Convergence Theorem

Let $(a_n)$ be a sequence in $\mathbb{R}$. Suppose that $(a_n)$ is bounded and monotone. Then $(a_n)$ converges. In particular,

1. If $a_n$ is increasing, then
$$\lim_{n\to\infty} a_n = \sup A,$$
2. If $a_n$ is decreasing, then
$$\lim_{n\to\infty} a_n = \inf A,$$

where we define $A = \{a_n \,:\, n \in \mathbb{N}\}$.

**Theorem 5.37**

Consider the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

We have that:

1. $(a_n)$ is monotone increasing,
2. $(a_n)$ is bounded.

In particular $(a_n)$ is convergent.

**Definition 5.38:** Euler's Number

The Euler's number is defined as

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

## 5.6 Special limits

**Theorem 5.39**

Let $x \in \mathbb{R}$, with $x > 0$. Then

$$\lim_{n\to\infty} \sqrt[n]{x} = 1.$$

**Theorem 5.40**

Let $(a_n)$ be a sequence such that $a_n \to 0$. Then

$$\sin(a_n) \to 0, \quad \cos(a_n) \to 1.$$

**Theorem 5.41**

Suppose $(a_n)$ is such that $a_n \to 0$ and $a_n \neq 0$. Then,

$$\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = 1.$$

**Theorem 5.42**

Suppose $(a_n)$ is such that $a_n \to 0$ and $a_n \neq 0$. Then,

$$\lim_{n\to\infty} \frac{1-\cos(a_n)}{(a_n)^2} = \frac{1}{2}, \quad \lim_{n\to\infty} \frac{1-\cos(a_n)}{a_n} = 0.$$

**Proof**

*Step 1.* By Theorem 5.40 and Theorem 5.41, we have

$$\cos(a_n) \to 1, \quad \frac{\sin(a_n)}{a_n} \to 1.$$

Therefore

$$\begin{aligned}\frac{1-\cos(a_n)}{(a_n)^2} &= \frac{1-\cos(a_n)}{(a_n)^2}\,\frac{1+\cos(a_n)}{1+\cos(a_n)}\\ &= \frac{1-\cos^2(a_n)}{(a_n)^2}\,\frac{1}{1+\cos(a_n)}\\ &= \left(\frac{\sin(a_n)}{a_n}\right)^2 \frac{1}{1+\cos(a_n)} \longrightarrow 1\cdot\frac{1}{1+1} = \frac{1}{2}\,,\end{aligned}$$

where in the last line we use the Algebra of Limits.
*Step 2.* We have

$$\frac{1-\cos(a_n)}{a_n} = a_n\cdot\frac{1-\cos(a_n)}{(a_n)^2} \longrightarrow 0\cdot\frac{1}{2} = 0\,,$$

using Step 1 and the Algebra of Limits.

**Example 5.43**

**Question.** Prove that

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = 1\,. \tag{5.5}$$

**Solution.** By Theorem 5.41 with $a_n = 1/n$, we get

$$n\sin\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \longrightarrow 1\,.$$

**Example 5.44**

**Question.** Prove that

$$\lim_{n\to\infty} n^2\left(1-\cos\left(\frac{1}{n}\right)\right) = \frac{1}{2}\,. \tag{5.6}$$

**Solution.** By Theorem 5.42 with $a_n = 1/n$, we have

$$n^2\left(1-\cos\left(\frac{1}{n}\right)\right) = \frac{1-\cos\left(\frac{1}{n}\right)}{\frac{1}{n^2}} \longrightarrow \frac{1}{2}\,.$$

**Example 5.45**

**Question.** Prove that

$$\lim_{n\to\infty} \frac{n\left(1-\cos\left(\frac{1}{n}\right)\right)}{\sin\left(\frac{1}{n}\right)} = \frac{1}{2}\,.$$

**Solution.** Using (5.6)-(5.5) and the Algebra of Limits

$$\begin{aligned}\frac{n\left(1-\cos\left(\frac{1}{n}\right)\right)}{\sin\left(\frac{1}{n}\right)} &= \frac{n^2\left(1-\cos\left(\frac{1}{n}\right)\right)}{n\sin\left(\frac{1}{n}\right)}\\ &\longrightarrow \frac{1/2}{1} = \frac{1}{2}\,.\end{aligned}$$

**Example 5.46**

**Question.** Prove that

$$\lim_{n\to\infty} n\cos\left(\frac{2}{n}\right)\sin\left(\frac{2}{n}\right) = 2\,.$$

**Solution.** We have

$$\cos\left(\frac{2}{n}\right) \longrightarrow 1\,,$$

by Theorem 5.40 applied with $a_n = 2/n$. Moreover

$$\frac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}} \longrightarrow 1\,,$$

by Theorem 5.41 applied with $a_n = 2/n$. Therefore

$$\begin{aligned}n\cos\left(\frac{2}{n}\right)\sin\left(\frac{2}{n}\right) &= 2\cdot\cos\left(\frac{2}{n}\right)\cdot\frac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}}\\ &\longrightarrow 2\cdot 1\cdot 1 = 2\,,\end{aligned}$$

where we used the Algebra of Limits.

**Example 5.47**

**Question.** Prove that

$$\lim_{n\to\infty} \frac{n^2+1}{n+1}\sin\left(\frac{1}{n}\right) = 1\,.$$

**Solution.** Using (5.5) and the Algebra of Limits,

$$\begin{aligned}\frac{n^2+1}{n+1}\sin\left(\frac{1}{n}\right) &= \left(\frac{1+\frac{1}{n^2}}{1+\frac{1}{n}}\right)\cdot\left(n\sin\left(\frac{1}{n}\right)\right)\\ &\longrightarrow \frac{1+0}{1+0}\cdot 1 = 1\,.\end{aligned}$$

# 6 Sequences in $\mathbb{C}$

**Definition 6.1:** Convergent sequence in $\mathbb{C}$

We say that a sequence $(a_n)$ in $\mathbb{C}$ **converges** to $a \in \mathbb{C}$, or equivalently has limit $a$, denoted by

$$\lim_{n\to\infty} a_n = a \quad \text{or} \quad a_n \to a\,,$$

if it holds:

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ s.t. } \forall\, n \geq N\,,\ |a_n - a| < \varepsilon\,.$$

If there exists $a \in \mathbb{C}$ such that $\lim_{n\to\infty} a_n = a$, we say that the sequence $(a_n)$ is **convergent**.

**Example 6.2**

**Question.** Using Definition 6.1, prove that

$$\lim_{n\to\infty} \frac{(3+i)n - 7i}{n} = 3 + i\,.$$

**Solution.**
*Part 1. Rough Work.* Let $\varepsilon > 0$. We need to clarify for which values of $n$ the following holds:

$$\left| \frac{(3+i)n - 7i}{n} - (3+i) \right| < \varepsilon\,.$$

We have

$$\left| \frac{(3+i)n - 7i}{n} - (3+i) \right| = \frac{|-7i|}{n} = \frac{7}{n}\,.$$

Therefore

$$\frac{7}{n} < \varepsilon \quad \iff \quad n > \frac{7}{\varepsilon}\,.$$

*Part 2. Formal Proof.* We want to prove that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \frac{(3+i)n - 7i}{n} - (3+i) \right| < \varepsilon\,, \qquad \forall\, n \geq N\,.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{7}{\varepsilon}\,.$$

The above is equivalent to

$$\frac{7}{N} < \varepsilon\,.$$

For $n \geq N$ we have

$$\left| \frac{(3+i)n - 7i}{n} - (3+i) \right| = \frac{7}{n} \leq \frac{7}{N} < \varepsilon\,.$$

**Definition 6.3:** Bounded sequence in $\mathbb{C}$

A sequence $(a_n)$ in $\mathbb{C}$ is called **bounded** if there exists a constant $M \in \mathbb{R}$, with $M > 0$, such that

$$|a_n| \leq M\,, \quad \forall\, n \in \mathbb{N}\,.$$

**Theorem 6.4**

If a sequence $(a_n)$ in $\mathbb{C}$ converges, then the sequence is bounded.

**Definition 6.5:** Divergent sequences in $\mathbb{C}$

We say that a sequence $(a_n)$ in $\mathbb{C}$ is **divergent** if it is not convergent.

**Corollary 6.6**

Let $(a_n)$ be a complex sequence. If $(a_n)$ is not bounded, then it is divergent.

## 6.1 Algebra of limits in $\mathbb{C}$

**Theorem 6.7:** Algebra of limits in $\mathbb{C}$

Let $(a_n)$ and $(b_n)$ be sequences in $\mathbb{C}$. Suppose that

$$\lim_{n\to\infty} a_n = a\,, \quad \lim_{n\to\infty} b_n = b\,,$$

for some $a, b \in \mathbb{C}$. Then,

1. Limit of sum is the sum of limits:
$$\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$$
2. Limit of product is the product of limits:
$$\lim_{n\to\infty} (a_n b_n) = ab$$
3. If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then
$$\lim_{n\to\infty} \left( \frac{a_n}{b_n} \right) = \frac{a}{b}$$

**Example 6.8**

**Question.** Compute the limit of

$$a_n = \frac{(2-i)n^2 + 6in - 5 - 3i}{(6+3i)n^2 + 11i}\,.$$

**Solution.** Factor $n^2$, the largest power of $n$ in the denominator,

$$a_n = \frac{(2-i) + \frac{6i}{n} - \frac{5}{n^2} - \frac{3i}{n^2}}{(6+3i) + \frac{11i}{n^2}} \longrightarrow \frac{2-i}{6+3i},$$

where we used the Algebra of Limits. Finally,

$$\frac{2-i}{6+3i} = \frac{(2-i)(6-3i)}{(6+3i)(6-3i)} = \frac{1}{5} - \frac{4}{15}i.$$

## 6.2 Convergence to zero

**Theorem 6.9**

Let $(a_n)$ be a sequence in $\mathbb{C}$ and suppose that

$$\lim_{n\to\infty} |a_n| = 0.$$

Then

$$\lim_{n\to\infty} a_n = 0.$$

**Example 6.10**

**Question.** Prove that $a_n \to 0$, where

$$a_n = \left(\frac{1}{2} + \frac{1}{3}i\right)^n.$$

**Solution.** We have

$$\begin{aligned} |a_n| &= \left|\left(\frac{1}{2} + \frac{1}{3}i\right)^n\right| \\ &= \left|\frac{1}{2} + \frac{1}{3}i\right|^n \\ &= \left(\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2}\right)^n \\ &= \left(\sqrt{\frac{13}{36}}\right)^n. \end{aligned}$$

Since

$$\left|\sqrt{\frac{13}{36}}\right| < 1,$$

by the Geometric Sequence Test for real sequences, we conclude that

$$|a_n| \to 0.$$

Hence $a_n \to 0$ by Theorem 6.9.

**Example 6.11**

**Question.** Consider the sequence

$$a_n := \frac{2i\cos(3n)n + (7-i)n^2}{3n^2 + 2in + \sin(2n)}.$$

Prove that

$$\lim_{n\to\infty} a_n = \frac{7}{3} - \frac{1}{3}i.$$

**Solution.** We divide by the largest power in the denominator, to get

$$a_n = \frac{\frac{2i\cos(3n)}{n} + (7-i)}{3 + \frac{2i}{n} + \frac{\sin(2n)}{n^2}}.$$

Notice that

$$-1 \le \cos(3n) \le 1, \quad \forall n \in \mathbb{N},$$

and thus

$$-\frac{2}{n} \le \frac{2\cos(3n)}{n} \le \frac{2}{n}, \quad \forall n \in \mathbb{N}.$$

Since

$$-\frac{2}{n} \longrightarrow 0, \quad \frac{2}{n} \longrightarrow 0,$$

by the Squeeze Theorem we conclude that also

$$\frac{2\cos(3n)}{n} \to 0.$$

In particular we have shown that

$$\left|\frac{2i\cos(3n)}{n}\right| = \left|\frac{2\cos(3n)}{n}\right| \to 0.$$

Using Theorem 6.9 we infer

$$\frac{2i\cos(3n)}{n} \to 0.$$

Similarly,

$$-\frac{1}{n^2} \le \frac{\sin(2n)}{n^2} \le -\frac{1}{n^2}, \quad \forall n \in \mathbb{N}.$$

Since

$$-\frac{1}{n^2} \longrightarrow 0, \quad \frac{1}{n^2} \longrightarrow 0,$$

by the Squeeze Theorem we conclude

$$\frac{\sin(2n)}{n^2} \longrightarrow 0.$$

Finally, we have

$$\left|\frac{2i}{n}\right| = \frac{2}{n} \longrightarrow 0,$$

and therefore

$$\frac{2i}{n} \longrightarrow 0$$

by Theorem 6.9. Using the Algebra of Limits in $\mathbb{C}$ we conclude

$$a_n = \frac{\frac{2i\cos(3n)}{n} + (7-i)}{3 + \frac{2i}{n} + \frac{\sin(2n)}{n^2}} \longrightarrow \frac{0 + (7-i)}{3+0+0} = \frac{7}{3} - \frac{1}{3}i.$$

## 6.3 Geometric sequence Test and Ratio Test in $\mathbb{C}$

**Theorem 6.12:** Geometric sequence Test in $\mathbb{C}$

Let $x \in \mathbb{C}$ and let $(a_n)_{n\in\mathbb{N}}$ be the geometric sequence in $\mathbb{C}$ defined by

$$a_n := x^n .$$

We have:

1. If $|x| < 1$, then
$$\lim_{n\to\infty} a_n = 0 .$$
2. If $|x| > 1$, then sequence $(a_n)$ is unbounded, and hence divergent.

**Example 6.13**

**Question.** Prove that $a_n \to 0$, where

$$a_n = \frac{(-1+4i)^n}{(7+3i)^n} .$$

**Solution.** We first rewrite

$$a_n = \frac{(-1+4i)^n}{(7+3i)^n} = \left(\frac{-1+4i}{7+3i}\right)^n$$

Then, we compute

$$\begin{aligned} \left|\frac{-1+4i}{7+3i}\right| &= \frac{|-1+4i|}{|7+3i|} \\ &= \frac{\sqrt{(-1)^2+4^2}}{\sqrt{7^2+3^2}} \\ &= \frac{\sqrt{17}}{\sqrt{58}} \\ &= \sqrt{\frac{17}{58}} \\ &< 1 \end{aligned}$$

By the Geometric Sequence Test $a_n \to 0$.

**Example 6.14**

**Question.** Prove that $a_n$ diverges, where

$$a_n = \frac{(-5+12i)^n}{(3-4i)^n} .$$

**Solution.** We first rewrite

$$a_n = \frac{(-5+12i)^n}{(3-4i)^n} = \left(\frac{-5+12i}{3-4i}\right)^n .$$

We compute

$$\begin{aligned} \left|\frac{-5+12i}{3-4i}\right| &= \frac{|-5+12i|}{|3-4i|} \\ &= \frac{\sqrt{5^2+(-12)^2}}{\sqrt{3^2+(-4)^2}} \\ &= \frac{13}{5} \\ &> 1 . \end{aligned}$$

By the Geometric Sequence Test, the sequence $a_n$ diverges.

**Example 6.15**

**Question.** Prove that $a_n$ diverges, where

$$a_n = \exp\left(\frac{i\pi}{2}n\right) .$$

**Solution.** We have

$$|a_n| = \left|e^{\frac{i\pi}{2}n}\right| = 1 ,$$

and hence the Geometric Sequence Test cannot be applied. However, we can see that

$$a_n = (i, -1, -i, 1, i, -1, -i, 1, \dots) ,$$

that is, $a_n$ assumes only the values $\{i, -1, -i, 1\}$, and each of them is assumed infinitely many times. Therefore $a_n$ is oscillating, and thus divergent.

**Theorem 6.16:** Ratio Test in $\mathbb{C}$

Let $(a_n)$ be a sequence in $\mathbb{C}$ such that

$$a_n \neq 0 , \quad \forall\, n \in \mathbb{N} .$$

1. Suppose that the following limit exists:
$$L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| .$$
Then,
   - If $L < 1$ we have
   $$\lim_{n\to\infty} a_n = 0 .$$
   - If $L > 1$, the sequence $(a_n)$ is unbounded, and hence does not converge.
2. Suppose that there exists $N \in N$ and $L > 1$ such that
$$\left|\frac{a_{n+1}}{a_n}\right| \geq L , \quad \forall\, n \geq N .$$
Then the sequence $a_n$ is unbounded, and hence does not converge.

**Example 6.17**

**Question.** Study the convergence / divergence of the sequence

$$a_n = \frac{(4-3i)^n}{(2n)!} .$$

**Solution.** We compute

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(4-3i)^{n+1}}{(2(n+1))!}\frac{(2n)!}{(4-3i)^n}\right|$$
$$= \frac{|4-3i|^{n+1}}{|4-3i|^n} \cdot \frac{(2n)!}{(2n+2)!}$$
$$= \frac{|4-3i|}{(2n+2)(2n+1)}$$
$$= \frac{\sqrt{4^2+(-3)^2}}{(2n+2)(2n+1)}$$
$$= \frac{5}{(2n+2)(2n+1)}$$
$$= \frac{\frac{5}{n^2}}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} \longrightarrow L = 0\,.$$

Since $L = 0 < 1$, by the Ratio Test in $\mathbb{C}$ we infer $a_n \to 0$.

## 6.4 Convergence of real and imaginary part

**Theorem 6.18**

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. For $n \in \mathbb{N}$, let $a_n, b_n \in \mathbb{R}$ such that

$$z_n = a_n + b_n i\,.$$

Let $z = a + bi$, with $a, b \in \mathbb{R}$. Then

$$\lim_{n\to\infty} z_n = z \quad\Longleftrightarrow\quad \lim_{n\to\infty} a_n = a\,, \quad \lim_{n\to\infty} b_n = b.$$

**Example 6.19**

**Question.** Consider the complex sequence

$$z_n := \frac{\left(4n+3n^2 i\right)\left(2n^2+i\right)}{5n^4}\,.$$

Show that

$$\lim_{n\to\infty} z_n = \frac{6}{5}i\,.$$

**Solution.** We find the real and imaginary parts of $z_n$

$$z_n = \frac{\left(4n+3n^2 i\right)\left(2n^2+i\right)}{5n^4}$$
$$= \frac{8n^3+4ni+6n^4 i+3n^2 i^2}{5n^4}$$
$$= \frac{8n^3-3n^2}{5n^4} + \frac{6n^4+4n}{5n^4}i$$
$$= a_n + b_n i\,.$$

Using the Algebra of Limits for real sequences we have that

$$a_n = \frac{8n^3-3n^2}{5n^4} = \frac{\frac{8}{n}-\frac{3}{n^2}}{5} \longrightarrow \frac{0-0}{5} = 0\,,$$
$$b_n = \frac{6n^4+4n}{5n^4} = \frac{6+\frac{4}{n^3}}{5} \longrightarrow \frac{6+0}{5} = \frac{6}{5}\,.$$

By Theorem 6.18 we conclude

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} a_n + i \lim_{n\to\infty} b_n = 0 + \frac{6}{5}i = \frac{6}{5}i\,.$$

# 7 Series

**Definition 7.1:** Partial sums

Let $(a_n)$ be a sequence in $\mathbb{C}$. The $k$-th partial sum of $(a_n)$ is

$$s_k := a_1 + a_2 + \ldots + a_k = \sum_{n=1}^{k} a_n$$

This sequence $(s_k)_{k\in\mathbb{N}}$ is called the sequence of **partial sums**.

**Definition 7.2:** Convergent series

Let $(a_n)$ be a sequence in $\mathbb{C}$. We denote the series of $(a_n)_{n\in\mathbb{N}}$ by

$$\sum_{n=1}^{\infty} a_n$$

We say that this series **converges** to $s \in \mathbb{C}$ if

$$\lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} s_k = s\,.$$

In this case we write

$$\sum_{n=1}^{\infty} a_n = s\,.$$

**Definition 7.3:** Divergent series

Let $(a_n)$ be a sequence in $\mathbb{C}$. The series

$$\sum_{n=1}^{\infty} a_n$$

is **divergent** if the sequence of partial sums $(s_k)$ is divergent.

**Example 7.4**

**Question.** Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1\,.$$

**Solution.** The idea to prove convergence is to split the general term into the sum of two fraction:

$$\begin{aligned} \frac{1}{n(n+1)} &= \frac{A}{n} + \frac{B}{n(n+1)} \\ &= \frac{A(n+1)+Bn}{n(n+1)} \\ &= \frac{(A+B)n + A}{n(n+1)}\,. \end{aligned}$$

In order for the LHS and RHS to be the same, we need to impose

$$(A+B)n + A = 1\,,$$

which holds if and only if

$$A + B = 1,\ A = 1 \quad \Longrightarrow \quad A = 1,\ B = -1\,.$$

Therefore, we conclude that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}\,.$$

We can now compute the partial sums $s_k$ as follows:

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{k} - \frac{1}{k+1} \\ &= 1 - \frac{1}{k+1}. \end{aligned}$$

Therefore,

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1\,,$$

which means that the series converges to 1, that is,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1\,.$$

A series of this kind is called a **telescopic series**, since we can *fold* the entire partial sum together, in such a way that only two terms remain.

**Example 7.5**

**Question.** Prove that the following series diverges

$$\sum_{n=1}^{\infty} (-1)^n\,.$$

**Solution.** The partial sums $s_k$ are given by

$$s_k = \sum_{n=1}^{k} (-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Therefore $s_k$ diverges, so also the series $\sum (-1)^n$ diverges.

**Theorem 7.6:** Necessary Condition for Convergence

Let $(a_n)$ be a sequence in $\mathbb{C}$. If the series

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$\lim_{n\to\infty} a_n = 0 \,.$$

**Example 7.7**

Consider the series

$$\sum_{n=1}^{\infty} (-1)^n \,. \tag{7.1}$$

We have that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \neq 0 \,,$$

being $(a_n)$ divergent. Therefore the series at (7.1) diverges by Theorem 7.6.

**Example 7.8**

**Question.** Discuss converge/divergence for the following series

$$\sum_{n=1}^{\infty} \frac{n}{5n+11} \,.$$

**Solution.** We have

$$a_n := \frac{n}{5n+11} = \frac{1}{5+\frac{11}{n}} \longrightarrow \frac{1}{5} \neq 0 \,.$$

Hence, the series $\sum a_n$ diverges.

**Important**

Theorem 7.6 says that if $\sum_{n=1}^{\infty} a_n$ converges, then

$$a_n \to 0 \,.$$

The converse is false: In general the condition $a_n \to 0$ does not guarantee convergence of the associated series, as shown in the example below.

**Example 7.9**

**Question.** Discuss convergence/divergence for the following series

$$\sum_{n=1}^{\infty} a_n \,, \quad a_n := \frac{1}{\sqrt{n+1}+\sqrt{n}} \,.$$

**Solution.** By the Algebra of Limits we have

$$\lim_{n\to\infty} a_n = 0 \,.$$

Therefore, we cannot conclude anything yet: The series might converge or diverge. Let us compute the partial sums:

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \frac{1}{\sqrt{k+1}+\sqrt{k}} \\ &= \sum_{n=1}^{k} \frac{1}{\sqrt{k+1}+\sqrt{k}} \cdot \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}} \\ &= \sum_{n=1}^{k} \sqrt{k+1}-\sqrt{k} \\ &= \sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\ldots+\sqrt{k+1}-\sqrt{k} \\ &= \sqrt{k+1}-1 \,. \end{aligned}$$

We have shown that the partial sums are

$$s_k = \sum_{n=1}^{k} a_n = \sqrt{k+1}-1 \,.$$

Therefore $(s_k)$ is divergent, and so the series $\sum a_n$ is divergent.

**Remark 7.10**

It is customary to sum a series starting at $n = 1$. However one could start the sum at any $n = N$ with $N \in \mathbb{N}$. This does not affect the convergence of the series, in the sense that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=N}^{\infty} a_n \text{ converges.}$$

In case of convergence, we would of course have

$$\sum_{n=N}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - (a_1 + \ldots + a_{N-1}) \,.$$

**Example 7.11**

**Question.** Prove that

$$\sum_{n=7}^{\infty} \frac{1}{n(n+1)} = \frac{1}{7} \,.$$

**Solution.** We have seen that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \,.$$

Hence also the series

$$\sum_{n=7}^{\infty} \frac{1}{n(n+1)}$$

converges. In this case, the partial sums are given by

$$\begin{aligned} s_k &= \sum_{n=7}^{k} \frac{1}{n(n+1)} \\ &= \sum_{n=7}^{k} \left( \frac{1}{n} - \frac{1}{n+1} \right) \\ &= \frac{1}{7} - \frac{1}{8} + \frac{1}{8} - \frac{1}{9} + \ldots + \frac{1}{k} - \frac{1}{k+1} \\ &= \frac{1}{7} - \frac{1}{k+1}\,. \end{aligned}$$

Therefore

$$\sum_{n=7}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} s_k = \frac{1}{7}\,.$$

## 7.1 Geometric series

**Definition 7.12:** Geometric Series in $\mathbb{C}$

Let $x \in \mathbb{C}$. The **geometric series** of ratio $x$ is the series

$$\sum_{n=0}^{\infty} x^n\,.$$

**Theorem 7.13:** Geometric Series Test

Let $x \in \mathbb{C}$. We have:

1. If $|x| < 1$, then the geometric series of ratio $x$ converges, with

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,. \tag{7.2}$$

2. If $|x| \geq 1$, then the geometric series of ratio $x$ diverges.

**Example 7.14**

**Question.** Discuss convergence/divergence of the following series. If the series converges, compute the limit.

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n, \qquad \sum_{n=0}^{\infty} \left(-\frac{3}{2}\right)^n,$$

$$\sum_{n=0}^{\infty} \left(\frac{-3}{4}\right)^n, \qquad \sum_{n=0}^{\infty} (-1)^n\,.$$

**Solution.**

1. Since $\left|\frac{1}{2}\right| < 1$, by the GST we have

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

2. Since $\left|\frac{-3}{2}\right| = \frac{3}{2} > 1$, by the GST the series

$$\sum_{n=0}^{\infty} \left(-\frac{3}{2}\right)^n$$

diverges.

3. Since $\left|\frac{-3}{4}\right| = \frac{3}{4} < 1$, we have

$$\sum_{n=0}^{\infty} \left(\frac{-3}{4}\right)^n = \frac{1}{1-\frac{-3}{4}} = \frac{1}{\frac{7}{4}} = \frac{4}{7}$$

4. Since $|-1| = 1$, the series

$$\sum_{n=0}^{\infty} (-1)^n$$

diverges.

**Remark 7.15**

If the sum of a Geometric Sries does not start at $n = 0$, we need to tweak the summation formula at (7.2). For example, if $|x| < 1$, and we start the series at $n = 1$, we get

$$\sum_{n=1}^{\infty} x^k = \frac{1}{1-x} - 1 = \frac{x}{1-x}\,.$$

**Example 7.16**

**Question.** Prove that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1\,.$$

**Solution.** We have that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - 1 \\ &= \frac{1}{1-\frac{1}{2}} - 1 = 1\,. \end{aligned}$$

**Example 7.17**

**Question.** Discuss convergence/divergence of the following series. If the series converges, compute the limit.

$$\sum_{n=0}^{\infty} \frac{1}{(1+i)^n}, \quad \sum_{n=0}^{\infty} \left(\frac{1-5i}{3+3i}\right)^n, \quad \sum_{n=0}^{\infty} \left(\frac{2+i}{3-2i}\right)^n.$$

**Solution.**

1. We have

$$\frac{1}{(1+i)^n} = \left(\frac{1}{1+i}\right)^n$$

and

$$\left|\frac{1}{1+i}\right| = \frac{1}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}} < 1\,.$$

Therefore, the series converges by the Geometric Series Test, and

$$\sum_{n=0}^{\infty} \frac{1}{(1+i)^n} = \frac{1}{1-\dfrac{1}{1+i}} = 1-i\,.$$

2. Since

$$\begin{aligned}\left|\frac{1-5i}{3+3i}\right| &= \frac{|1-5i|}{|3+3i|} \\ &= \frac{\sqrt{(1)^2+(-5)^2}}{3\sqrt{1^2+1^2}} \\ &= \frac{\sqrt{26}}{3\sqrt{2}} \\ &= \frac{\sqrt{13}}{3} > 1\,,\end{aligned}$$

the series diverges by the Geometric Series Test.

3. We have

$$\begin{aligned}\left|\frac{2+i}{3-2i}\right| &= \frac{|2+i|}{|3-2i|} \\ &= \frac{\sqrt{2^2+1^2}}{\sqrt{3^2+(-2)^2}} \\ &= \sqrt{\frac{5}{13}} < 1\,.\end{aligned}$$

Therefore the series converges by the Geometric Series Test, and

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{2+i}{3-2i}\right)^n &= \frac{1}{1-\dfrac{2+i}{3-2i}} \\ &= \frac{1}{\dfrac{3-2i-(2+i)}{3-2i}} \\ &= \frac{3-2i}{1-3i} \\ &= \frac{3-2i}{1-3i}\,\frac{1+3i}{1+3i} \\ &= \frac{3-2i+9i-6i^2}{1-9i^2} \\ &= \frac{9}{10}+\frac{7}{10}i\end{aligned}$$

## 7.2 Algebra of Limits for Series

**Theorem 7.18:** Algebra of Limits for Series

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{C}$ and let $c\in\mathbb{C}$. Suppose that

$$\sum_{n=1}^{\infty} a_n = a\,, \qquad \sum_{n=1}^{\infty} b_n = b\,.$$

Then:

1. The sum of series is the series of the sums:

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = a \pm b\,.$$

2. The product of a series with a number obeys

$$\sum_{n=1}^{\infty} c\cdot a_n = c\cdot a\,.$$

**Example 7.19**

**Question.** Prove that

$$\sum_{n=0}^{\infty}\left(2\left(\frac{1}{3}\right)^n+\left(\frac{2}{3}\right)^n\right) = 6\,.$$

**Solution.** Note that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n &= \frac{1}{1-\dfrac{1}{3}} = \frac{3}{2}\,, \\ \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n &= \frac{1}{1-\dfrac{2}{3}} = 3\,,\end{aligned}$$

by the Geometric Series Test. Therefore, we can apply the Algebra of Limit for Series to conclude that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(2\left(\frac{1}{3}\right)^n+\left(\frac{2}{3}\right)^n\right) &= 2\cdot\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n+\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n \\ &= 2\cdot\frac{3}{2}+3 = 6\end{aligned}$$

## 7.3 Non-negative series

**Definition 7.20:** Non-negative series

Let $(a_n)$ be a sequence in $\mathbb{R}$. We call the series

$$\sum_{n=1}^{\infty} a_n$$

a **non-negative series** if

$$a_n \geq 0\,, \quad \forall\, n\in\mathbb{N}\,.$$

**Lemma 7.21**

Let $(a_n)$ be a sequence in $\mathbb{R}$ with

$$a_n \geq 0\,, \quad \forall\, n\in\mathbb{N}\,.$$

Define the partial sums as

$$s_k := \sum_{n=1}^{k} a_n\,.$$

The sequence $(s_k)$ is increasing.

We present 4 test for the convergence of non-negative series:

1. Cauchy Condensation Test
2. Comparison Test
3. Limit Comparison Test
4. Ratio Test (positive series only)

**Theorem 7.22:** Cauchy Condensation Test

Let $(a_n)$ be a sequence in $\mathbb{R}$. Suppose that $(a_n)$ is non-negative and decreasing, that is,

$$a_n \geq a_{n+1}\,, \quad \forall\, n \in \mathbb{N}\,.$$

They are equivalent:

1. The following series converges

$$\sum_{n=1}^{\infty} a_n\,.$$

2. The following series converges

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 8a_8 + 16a_{16} + \dots$$

**Theorem 7.23:** Convergence of $p$-series

Let $p \in \mathbb{R}$. Consider the $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}\,.$$

We have:

1. If $p > 1$ the $p$-series converges.
2. If $p \leq 1$ the $p$-series diverges.

**Proof**

The series in question is

$$\sum_{n=1}^{\infty} a_n\,, \quad a_n := \frac{1}{n^p}\,.$$

Note that $(a_n)$ is decreasing and non-negative. Hence, by the Cauchy Condensation Test of Theorem 7.22, the $p$-series converges if and only if

$$\sum_{n=0}^{\infty} 2^n a_{2^n}$$

converges. We have

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^{n-np} = \sum_{n=0}^{\infty} (2^{1-p})^n\,,$$

and the latter is a Geometric Series of ratio

$$x := 2^{1-p}\,.$$

By the Geometric Series Test, we have convergence if and only if

$$|x| < 1\,,$$

which is equivalent to

$$\begin{aligned} 2^{1-p} < 1 = 2^0 \quad &\iff \quad 1 - p < 0 \\ &\iff \quad p > 1\,. \end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$, ending the proof.

**Theorem 7.24**

Let $p \in \mathbb{R}$. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}\,.$$

We have:

1. If $p > 1$ the series converges.
2. If $p \leq 1$ the series diverges.

**Proof**

The series in question is

$$\sum_{n=2}^{\infty} a_n\,, \quad a_n := \frac{1}{n\,(\log n)^p}\,.$$

Note that $(a_n)$ is non-negative and decreasing. Therefore we can apply the Cauchy Condensation Test to conclude that the above series is convergent if and only if the series

$$\sum_{n=1}^{\infty} 2^n a_{2^n}$$

is convergent. We have

$$2^n a_{2^n} = 2^n \frac{1}{2^n\,(\log 2^n)^p} = \frac{1}{n^p\,\log 2}$$

so that

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \frac{1}{\log 2} \sum_{n=1}^{\infty} \frac{1}{n^p}\,.$$

The latter is a $p$-series, which by Theorem 7.23 converges if and only if $p > 1$. Hence

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}$$

converges if and only if $p > 1$, and the proof is concluded.

**Theorem 7.25:** Comparison test

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be non-negative sequences. Suppose that there exists $N \in \mathbb{N}$ such that

$$a_n \leq b_n\,, \quad \forall\, n \geq N\,.$$

They hold:

$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges,}$$
$$\sum_{n=1}^{\infty} a_n \text{ diverges} \implies \sum_{n=1}^{\infty} b_n \text{ diverges.}$$

**Example 7.26**

**Question.** Discuss convergence/divergence of the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n-1}, \tag{7.3}$$

$$\sum_{n=0}^{\infty} \frac{3^n + 6n + \frac{1}{n+1}}{2^n}. \tag{7.4}$$

**Solution.**

1. Since $3n - 1 \geq 0$ for all $n \in \mathbb{N}$, we get

$$\frac{1}{n^2+3n-1} \leq \frac{1}{n^2}, \quad \forall n \in \mathbb{N}.$$

By Theorem 7.23 the $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. Therefore also the series at (7.3) converges by the Comparison Test in Theorem 7.25.

2. Note that

$$\frac{3^n + 6n + \frac{1}{n+1}}{2^n} \geq \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n, \quad \forall n \in \mathbb{N}.$$

Since $\left|\frac{3}{2}\right| = \frac{3}{2} > 1$, the series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$$

diverges by the Geometric Series Test in Theorem 7.13. Therefore, by the Comparison Test, also the series at (7.4) diverges.

**Theorem 7.27:** Limit Comparison Test

Let $(a_n)$ and $(b_n)$ be sequences such that

$$a_n \geq 0, \quad b_n > 0, \quad \forall n \in \mathbb{N}.$$

Suppose there exists $L \in \mathbb{R}$ such that

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

They hold:

1. If $0 < L < \infty$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges.}$$

2. If $L = 0$, then

$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges,}$$
$$\sum_{n=1}^{\infty} a_n \text{ diverges} \implies \sum_{n=1}^{\infty} b_n \text{ diverges.}$$

**Example 7.28**

**Question.** Prove that the following series converges

$$\sum_{n=1}^{\infty} \frac{2n^3+5n+1}{7n^6+2n+5}.$$

**Solution.** Set

$$a_n := \frac{2n^3+5n+1}{7n^6+2n+5}, \quad b_n := \frac{1}{n^3}.$$

We have

$$\begin{aligned} L &:= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{2n^3+5n+1}{7n^6+2n+5} \Big/ \frac{1}{n^3} \\ &= \lim_{n\to\infty} \frac{2n^6+5n^4+n^3}{7n^6+2n+5} \\ &= \lim_{n\to\infty} \frac{2 + \frac{5}{n^2} + \frac{1}{n^3}}{7 + \frac{2}{n^5} + \frac{5}{n^6}} = \frac{2}{7}. \end{aligned}$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges, being a $p$-series with $p = 3 > 1$. Since $L = \frac{2}{7} > 0$, also the series

$$\sum_{n=1}^{\infty} \frac{2n^3+5n+1}{7n^6+2n+5}$$

converges, by the Limit Comparison Test.

**Example 7.29**

**Question.** Prove that the following series diverges

$$\sum_{n=1}^{\infty} \frac{n+\cos(n)}{n^2}.$$

**Solution.** Since $\sin(n)$ is bounded, we expect the terms in the series to behave like $1/n$ for large $n$. Hence we set

$$a_n := \frac{n+\cos(n)}{n^2}, \quad b_n = \frac{1}{n}.$$

We compute

$$L := \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+\cos(n)}{n^2} \Big/ \frac{1}{n}$$
$$= \lim_{n\to\infty} \frac{n^2 + n\cos(n)}{n^2}$$
$$= \lim_{n\to\infty} \left(1 + \frac{\cos(n)}{n}\right)$$

Note that

$$-1 \leq \cos(n) \leq 1 \quad \Longrightarrow \quad -\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}.$$

As both $-\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$, by the Squeeze Theorem

$$\frac{\cos(n)}{n} \longrightarrow 0.$$

Hence

$$L = \lim_{n\to\infty} \left(1 + \frac{\cos(n)}{n}\right) = 1.$$

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Since $L = 1 > 0$, the series

$$\sum_{n=1}^{\infty} \frac{n+\cos(n)}{n^2}.$$

diverges by the Limit Comparison Test.

**Example 7.30**

**Question.** Prove that the following series converges

$$\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right).$$

**Solution.** Since

$$\cos\left(\frac{1}{n}\right) \leq 1,$$

the above is a non-negative series. Recall the limit

$$\lim_{n\to\infty} \frac{1-\cos(a_n)}{(a_n)^2} = \frac{1}{2},$$

where $(a_n)$ is a sequence in $\mathbb{R}$ such that $a_n \to 0$ and

$$a_n \neq 0 \quad \forall\, n \in \mathbb{N}.$$

In particular, for $a_n = 1/n$, we obtain

$$\lim_{n\to\infty} n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right) = \frac{1}{2}.$$

Set

$$b_n := 1 - \cos\left(\frac{1}{n}\right), \quad c_n := \frac{1}{n^2}.$$

We have

$$L := \lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right) = \frac{1}{2}.$$

Note that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, being a $p$-series with $p > 2$. Therefore, since $L = 1/2 > 0$, also the series

$$\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$$

converges, by the Limit Comparison Test.

**Example 7.31**

**Question.** Prove that the following series converges

$$\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}.$$

**Solution.** Since

$$\sin(n) \geq -1,$$

the above is a non-negative series. As $\sin(n)$ is bounded, the series behaves similarly to

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

However

$$\frac{1+\sin(n)}{n^2} \Big/ \frac{1}{n^2} = 1 + \sin(n)$$

does not converge. Hence, we cannot use the Limit Comparison Test. In alternative, we note that

$$\frac{1+\sin(n)}{n^2} \leq \frac{2}{n^2}, \quad \forall\, n \in \mathbb{N}.$$

The series

$$\sum_{n=1}^{\infty} \frac{2}{n^2}$$

converges, being a $p$-series with $p = 2 > 1$. Therefore also

$$\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$$

converges, by the Comparison Test of Theorem 7.25.

**Theorem 7.32:** Ratio Test for positive series

Let $(a_n)$ be a sequence in $\mathbb{R}$ such that

$$a_n > 0, \quad \forall\, n \in \mathbb{N}.$$

1. Suppose that the following limit exists:

$$L := \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

They hold:

- If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.
- If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

2. Suppose that there exists $N \in \mathbb{N}$ and $L > 1$ such that

$$\frac{a_{n+1}}{a_n} \geq L, \quad \forall\, n \geq N.$$

Then the series $\sum_{n=1}^{\infty} a_n$ diverges.

**Example 7.33**

**Question.** Discuss convergence/divergence of the following series

$$\sum_{n=1}^{\infty} a_n, \quad a_n = \frac{(n!)^2}{(2n)!}.$$

**Solution.** We compute

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{((n+1)!)^2}{(2(n+1))!}\Bigg/\frac{(n!)^2}{(2n)!}$$
$$= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)}$$
$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{1}{4}\,.$$

Since $L = 1/4 < 1$, by the Ratio Test we conclude that $\sum a_n$ converges.

**Example 7.34**

**Question.** Using the Cauchy Condensation Test and the Ratio Test, prove that the following series converges

$$\sum_{n=1}^{\infty}\frac{\log(n)}{n^2}\,.$$

**Solution.** Set $a_n = \log n/n^2$. By the Cauchy Condensation Test, we know that $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges. We have:

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^n \frac{\log(2^n)}{(2^n)^2}$$
$$= \log(2)\sum_{n=0}^{\infty}\frac{n}{2^n}$$
$$= \log(2)\sum_{n=0}^{\infty} b_n\,, \qquad b_n := \frac{n}{2^n}\,.$$

Apply the Ratio Test to the series $\sum b_n$

$$\frac{b_{n+1}}{b_n} = \frac{n+1}{2^{n+1}}\Bigg/\frac{n}{2^n} = \frac{n+1}{2n} \longrightarrow \frac{1}{2} < 1\,.$$

Therefore, $\sum b_n$ converges by the Ratio Test, so that also $\sum 2^n a_{2^n}$ converges. We conclude that $\sum a_n$ converges by the Cauchy Condensation Test.

## 7.4 General series

**Definition 7.35:** Absolute convergence

Let $(a_n)$ be a sequence in $\mathbb{C}$. The series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely** if the following non-negative series converges

$$\sum_{n=1}^{\infty}|a_n|\,.$$

**Theorem 7.36:** Absolute Convergence Test

Let $(a_n)$ be a sequence in $\mathbb{C}$. If the series $\sum_{n=1}^{\infty} a_n$ converge absolutely, then the series converges.

**Example 7.37**

**Question.** Discuss absolute convergence of the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\,.$$

**Solution.** The series does not converge absolutely, since

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

does not converge, being the harmonic series.

**Example 7.38**

**Question.** Prove that the following series converges

$$\sum_{n=1}^{\infty} a_n\,, \qquad a_n = (-1)^n\frac{n^2-5n+2}{n^4}\,.$$

**Solution.** We have

$$|a_n| = \frac{|n^2-5n+2|}{n^4} = \frac{n^2+5n+2}{n^4}\,,$$

for $n$ sufficiently large (e.g. $n \geq 10$). Note that

$$\frac{n^2+5n+2}{n^4}\Bigg/\frac{1}{n^2} = \frac{n^4+5n^3+2n^2}{n^4}$$
$$= 1+\frac{5}{n}+\frac{2}{n^2} \longrightarrow 1$$

The series $\sum 1/n^2$ converges, being a $p$-series with $p = 2$. Hence, also

$$\sum_{n=1}^{\infty}\frac{n^2+5n+2}{n^4}$$

converges, by the Limit Comparison Test for non-negative series (Theorem 7.27). This shows $\sum |a_n|$ converges, which means that $\sum a_n$ converges absolutely. In particular, $\sum a_n$ converges by the Absolute Convergence Test.

**Theorem 7.39:** Ratio Test for general series

Let $(a_n)$ be a sequence in $\mathbb{C}$, such that

$$a_n \neq 0 \quad \forall\, n \in \mathbb{N}\,.$$

1. Suppose that the following limit exists:

$$L := \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\,.$$

They hold:

- If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely, and hence converges.
- If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

2. Suppose that there exists $N \in \mathbb{N}$ and $L > 1$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \geq L\,, \quad \forall\, n \geq N\,.$$

Then the series $\sum_{n=1}^{\infty} a_n$ diverges.

**Example 7.40**

**Question.** Prove that the series converges

$$\sum_{n=1}^{\infty} a_n \,, \quad a_n = \frac{(4-3i)^n}{(n+1)!} \,.$$

**Solution.** We have

$$\begin{aligned} L &:= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(4-3i)^{n+1}}{((n+1)+1)!} \middle/ \frac{(4-3i)^n}{(n+1)!} \right| \\ &= \lim_{n\to\infty} \frac{5}{n+2} = 0 \,. \end{aligned}$$

As $L = 0 < 1$, we conclude that $\sum a_n$ converges absolutely, by the Ratio Test. Hence, $\sum a_n$ converges by the Absolute Convergence Test.

**Theorem 7.41:** Exponential series

Let $z \in \mathbb{C}$. The **exponential series**

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely.

**Proof**

Set $a_n = z^n/n!$. Then

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \middle/ \frac{z^n}{n!} \right| \\ &= \lim_{n\to\infty} \frac{|z|}{n+1} = 0 \,. \end{aligned}$$

Therefore the series converges absolutely by the Ratio Test in Theorem 7.39.

## 7.5 Conditional convergence

**Definition 7.42:** Conditional convergence

Let $(a_n)$ be a sequence in $\mathbb{C}$. We say that the series

$$\sum_{n=1}^{\infty} a_n$$

converges **conditionally** if it converges, but it does not converge absolutely.

**Definition 7.43:** Rearrangement of a series

Let $(a_n)$ be a sequence in $\mathbb{C}$. Then:

1. A **permutation** is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$.
2. A **rearrangement** of the series $\sum_{n=1}^{\infty} a_n$ is a series

$$\sum_{n=1}^{\infty} a_{\sigma(n)}$$

for some permutation $\sigma$.

**Theorem 7.44**

Let $(a_n)$ be a sequence in $\mathbb{C}$ such that

$$\sum_{n=1}^{\infty} |a_n|$$

converges. For any permutation $\sigma$ we have

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n \,.$$

**Theorem 7.45:** Riemann rearrangement Theorem

Let $(a_n)$ be a real sequence such that the series

$$\sum_{n=1}^{\infty} a_n$$

converges conditionally. Let

$$L \in \mathbb{R} \quad \text{or} \quad L = \pm\infty \,.$$

There exists a permutation $\sigma$ such that the corresponding rearrangement $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges conditionally to $L$, that is,

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = L \,.$$

**Theorem 7.46:** Dirichlet Test

Let $(c_n)$ be a sequence in $\mathbb{C}$ and $(q_n)$ a sequence in $\mathbb{R}$. Suppose that

- $q_n$ is decreasing,
- $q_n \to 0$,
- $q_n \geq 0$ for all $n \in \mathbb{N}$.
- Suppose there exists $M > 0$ such that

$$\left| \sum_{n=1}^{k} c_n \right| \leq M \,, \quad \forall\, k \in \mathbb{N} \,.$$

Then the following series converges

$$\sum_{n=1}^{\infty} c_n q_n \,.$$

**Example 7.47**

**Question.** Let $\theta \in \mathbb{R}$, with

$$\theta \neq 2k\pi \,, \quad \forall \, k \in \mathbb{Z} \,.$$

Prove that the below series are conditionally convergent

$$\sum_{n=1}^{\infty} \frac{e^{i\theta n}}{n} \,, \quad \sum_{n=1}^{\infty} \frac{\cos(\theta n)}{n} \,, \quad \sum_{n=1}^{\infty} \frac{\sin(\theta n)}{n} \,.$$

**Solution.**

1. Recalling the Euler's Identity

$$e^{i\theta} = \cos(\theta) + i \sin(\theta) \,,$$

we obtain that

$$\sum_{n=1}^{\infty} \frac{e^{i\theta n}}{n} = \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} + i \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} \,.$$

Therefore, the series $\sum e^{i\theta n}/n$ converge conditionally if and only if $\sum \cos(\theta n)/n$ and $\sum \sin(\theta n)/n$ converge conditionally. It is then sufficient to study $\sum e^{i\theta n}/n$.

2. The series $\sum e^{i\theta n}/n$ does not converge absolutely, since

$$\sum_{n=1}^{\infty} \left| \frac{e^{i\theta n}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, being the Harmonic Series.

3. Set $c_n = e^{i\theta n}$, $q_n = 1/n$, so that

$$\sum_{n=1}^{\infty} \frac{e^{i\theta n}}{n} = \sum_{n=1}^{\infty} c_n q_n \,.$$

We have that $q_n$ is decreasing, $q_n \to 0$ and $q_n \geq 0$. Let us prove that there exists $M > 0$ such that

$$\left| \sum_{n=1}^{k} e^{i\theta n} \right| \leq M \,, \quad \forall \, k \in \mathbb{N} \,. \tag{7.5}$$

Note that

$$1 - e^{i\theta} \neq 0 \,,$$

since $\theta \neq 2k\pi$ for all $k \in \mathbb{Z}$. Therefore we can use the Geometric Series (truncated) summation formula to get

$$\begin{aligned} \sum_{n=1}^{k} e^{i\theta n} &= \sum_{n=1}^{k} (e^{i\theta})^n \\ &= \frac{1 - e^{i(k+1)\theta}}{1 - e^{i\theta}} - 1 \\ &= e^{i\theta} \, \frac{1 - e^{ik\theta}}{1 - e^{i\theta}} \end{aligned}$$

Taking the modulus

$$\begin{aligned} \left| \sum_{n=1}^{k} e^{i\theta n} \right| &= \left| e^{i\theta} \, \frac{1 - e^{ik\theta}}{1 - e^{i\theta}} \right| = \left| e^{i\theta} \right| \left| \frac{1 - e^{ik\theta}}{1 - e^{i\theta}} \right| \\ &= \frac{|1 - e^{ik\theta}|}{|1 - e^{i\theta}|} \leq \frac{|1| + |e^{ik\theta}|}{|1 - e^{i\theta}|} = \frac{2}{|1 - e^{i\theta}|} \,, \end{aligned}$$

where we used the triangle inequality. Since the RHS does not depend on $k$, we can set

$$M = \frac{2}{|1 - e^{i\theta}|} \,,$$

so that (7.5) holds. Therefore, $\sum e^{i\theta n}/n$ converges by the Dirichlet Test.

4. We have shown that $\sum e^{i\theta n}/n$ converges, but not absolutely. Hence, it converges conditionally.

**Theorem 7.48:** Alternate Convergence Test

Let $(q_n)$ be a sequence in $\mathbb{R}$ such that

- $q_n$ is decreasing,
- $q_n \to 0$,
- $q_n \geq 0$ for all $n \in \mathbb{N}$.

The following series converges

$$\sum_{n=1}^{\infty} (-1)^n q_n$$

**Example 7.49**

**Question.** Prove that the series converges conditionally

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \,.$$

**Solution.** The series does not converge absolutely, since

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, being the Harmonic Series. Set $q_n = 1/n$, so that

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = \sum_{n=1}^{\infty} (-1)^n q_n \,.$$

Clearly, $q_n \geq 0$, $q_n \to 0$ and $q_n$ is decreasing. Hence, the series converges by the Alternating Series Test. Thus, the series converges conditionally.

**Theorem 7.50:** Abel's Test

Let $(a_n)$ and $(q_n)$ be sequences in $\mathbb{R}$. Suppose that

- $q_n$ is monotone and bounded,
- The series $\sum a_n$ converges.

Then the following series converges

$$\sum_{n=1}^{\infty} a_n q_n \,.$$

**Example 7.51**

**Question.** Prove that the series converges conditionally

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(1 + \frac{1}{n}\right)^n .$$

**Solution.** Set

$$a_n := \frac{(-1)^n}{n}, \quad q_n := \left(1 + \frac{1}{n}\right)^n .$$

We have seen that $q_n$ is monotone increasing and bounded (recall that $q_n \to e$). Moreover, the series $\sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test, as seen in Example 7.49. Hence the series $\sum_{n=1}^{\infty} a_n q_n$ converges by the Abel Test.
However, the series in question does not converge absolutely. Indeed,

$$\left|\frac{(-1)^n}{n} \left(1 + \frac{1}{n}\right)^n\right| = \frac{1}{n} q_n \geq \frac{1}{n} q_1 = \frac{2}{n},$$

since $(q_n)$ is increasing. As the series $\sum 2/n$ diverges, by the Comparison Test we conclude that also

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n} \left(1 + \frac{1}{n}\right)^n\right|$$

diverges. Therefore, the series in the example converges conditionally.

**Good Luck with the Exam!**

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